Quadratic Forms

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The history of concerning quadratic forms is backed to ancient Greece and India, it is of greatest importance due to its connection to the shape of orbits of planets.

- Representation of integers as sums of two squares, motivated by Pythagoras theorem and geometry.
- Solution of Pell's equation. Pell's equation was considered by the Indian mathematician Brahmagupta in the 7th century CE. The Pell's equation, or called Pell-Fermat equation is

$$x^2 - ny^2 = 1$$

  where n is a given positive nonsquare integer, and integer solutions are sought for x and y.

- Specific Pell's equation was considered 400 BC in Greece. The positive integer solutions for
$$x^2 - 2y^2 = 1 \quad \text{or } x^2 - ny^2 = -1$$
can be used to approximate $\sqrt{2}$, i.e., using $\frac{x}{y}$. For example, $x = 17, y = 12$ and $x = 577, y = 408$ will give two approximations, $17/12$ and $577/408$, of $\sqrt{2}$.
- Lagrange proved that, as long as n is not a perfect square, Pell's equation has infinitely many distinct integer solutions. Actually, William Brouncker was the first European to solve this equation, but Euler mistakenly attributed Brouncker's solution to John Pell.
- In general, a binary quadratic form is a quadratic homegeneous polynomial in two variables
$$Q(x, y) = ax^2 + bxy + cy^2$$
the coefficients, of course, can go beyond integral numbers.

- Taylor's theorem provides a way to approximate a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ near a point $\mathbf{x}_0$ using polynomials. The expansion to second order is

$$f(\mathbf{x}_0 + \mathbf{v}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)^\top \mathbf{v} + \frac{1}{2}\mathbf{v}^\top \nabla^2 f(\mathbf{x}_0)\mathbf{v} + o(\|\mathbf{v}\|^2)$$

where $\nabla^2 f(\mathbf{x}_0)$ is the Hessian matrix with ij-entries

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0)$$

- The Hessian matrix defines the quadratic form $\mathbf{v}^\top H(\mathbf{x}_0)\mathbf{v}$ in the vector space with $\mathbf{x}_0$ as the origin. The positive and negative definite property encodes the local minimum or local maximum of f at $\mathbf{x}_0$.

Conics and Quadratic Surfaces

In this part, we will use the spectrual theorem to analyze the equations of conic sections and quadratic surfaces. Suppose we are given the quadratic equation

$$x_1^2 + 4x_1x_2 - 2x_2^2 = 6$$

Notice that we can write the quadratic expression

$$x_1^2 + 4x_1x_2 - 2x_2^2 = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}^\top A\mathbf{x}$$

where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

is the symmetric matrix. To diagonalize A, we have to finish following three steps:

1. Find the eigenvalues of A. Set

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} == \lambda^2 + \lambda - 6 = 0$$

The eigenvalues are

$$\lambda_1 = 2, \quad \lambda_2 = -3.$$

2. Find the eigenvectors.

For $\lambda_1 = 2$,

$$\begin{bmatrix} 1-2 & 2 \\ 2 & -2-2 \end{bmatrix} \mathbf{v} = \mathbf{0}$$

Then

$$-v_1 + 2v_2 = 0,$$

and an eigenvector corresponding to $\lambda_1 = 2$ is $(2, 1)^\top$.

For $\lambda_2 = -3$,

$$\begin{bmatrix} 1-(-3) & 2 \\ 2 & -2-(-3) \end{bmatrix} \mathbf{v} = \mathbf{0}$$

gives

$$4v_1 + 2v_2 = 0$$

and then an eigenvector corresponding to $\lambda_2 = -3$ is $(-1, 2)^\top$
Thus, we have that

$$A = Q\Lambda Q^\top$$

where

$$Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \quad \text{and } \Lambda = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}.$$

If we make the substitution $\mathbf{y} = Q^\top \mathbf{x}$, then we have

$$\mathbf{x}^\top A\mathbf{x} = \mathbf{x}^\top (Q\Lambda Q^\top)\mathbf{x} = (Q^\top \mathbf{x})^\top \Lambda (Q^\top \mathbf{x}) = \mathbf{y}^\top \Lambda \mathbf{y} = 2y_1^2 - 3y_2^2.$$

Note that the conic is much easier to understand in the $y_1 y_2$-coordinates.

The same trick can be used for any quadratic equation

$$\alpha x_1^2 + 2\beta x_1 x_2 + \gamma x_2^2 = \delta,$$

where $\alpha, \beta, \gamma, \delta$ are real numbers. Now we set

$$A = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix},$$

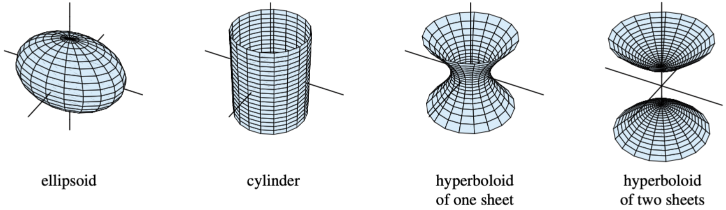
so that the equation can be written as

$$\mathbf{x}^\top A \mathbf{x} = \delta.$$

Since A is symmetric, we can find a diagonal matrix Λ and an orthogonal matrix Q so that $A = Q\Lambda Q^\top$. Thus, setting $\mathbf{y} = Q^\top \mathbf{x}$, we can rewrite equation as

$$\mathbf{y}^\top \Lambda \mathbf{y} = \lambda_1 y_1^2 + \lambda_2 y_2^2 = \delta.$$

From this expression, we can infer that the "shape" of a quadratic form as a two-variable function is determined by the signs of eigenvalues of A. Quadratic surfaces include those shown in the following figure.



Consider the surface defined by the equation

$$2x_1x_2 + 2x_1x_3 + x_2^2 + x_3^2 = 2.$$

Observe that if

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

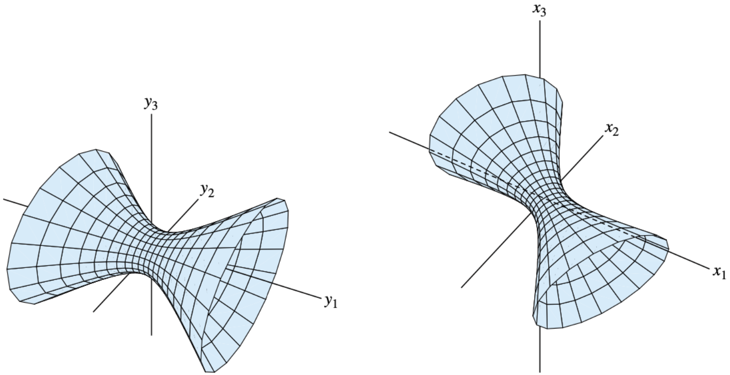
is the symmetric matrix, then

$$\mathbf{x}^\top A\mathbf{x} = 2x_1x_2 + 2x_1x_2 + x_2^2 + x_3^2,$$

and so we use the diagonalization and the substitution $\mathbf{y} = Q^\top \mathbf{x}$ as before to write

$$\mathbf{x}^\top A\mathbf{x} = \mathbf{y}^\top \Lambda \mathbf{y}, \quad \text{where } \Lambda = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

From the **y**-coordinates we can see that the graph is the hyperboloid of one sheet.



Quadratic Forms

Definition

A *quadratic form* in n variables $x_1, ..., x_n$ is a homogeneous second-degree polynomial in these variables. So the polynomial has the form

$$Q(x_1, ..., x_n) = \sum_{i,j}^{n} q_{ij} x_i x_j.$$

Every quadratic form can be viewed as a function of the vector $\mathbf{x} = x_1\mathbf{v}_1 + ... + x_n\mathbf{v}_n$, where $\mathbf{v}_1, ..., \mathbf{v}_n$ is some fixed basis of the vector space V of degree n.

Definition

A function $B(\mathbf{x}, \mathbf{y})$ that assigns to two vectors $\mathbf{x}, \mathbf{y} \in V$ a scalar value is called a *bilinear form* on V if it is linear in each of its arguments. In other words, the following conditions must be satisfied for all vectors of the space V and scalars c.

$$\begin{aligned} B(\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}) &= B(\mathbf{x}_1, \mathbf{y}) + B(\mathbf{x}_2, \mathbf{y}) \\ B(c\mathbf{x}, \mathbf{y}) &= cB(\mathbf{x}, \mathbf{y}) \\ B(\mathbf{x}, \mathbf{y}_1 + \mathbf{y}_2) &= B(\mathbf{x}, \mathbf{y}_1) + B(\mathbf{x}, \mathbf{y}_2) \\ B(\mathbf{x}, c\mathbf{y}) &= cB(\mathbf{x}, \mathbf{y}). \end{aligned}$$

If $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is some basis of V, then we can express the bilinear form in terms of the coordinates of the vectors

$$B(\mathbf{x}, \mathbf{y}) = \sum_{i,j=1}^{n} = b_{ij} x_i y_i, \quad \text{where } b_{ij} = B(\mathbf{v}_i, \mathbf{v}_j).$$

In this case, the square matrix $B = (b_{ij})$ is called the matrix of the bilinear form B in the basis $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$.

Definition

A bilinear form $B(\mathbf{x}, \mathbf{y})$ is said to be *symmetric* if

$$B(\mathbf{x}, \mathbf{y}) = B(\mathbf{y}, \mathbf{x})$$

and *antisymmetric* if

$$B(\mathbf{x}, \mathbf{y}) = -B(\mathbf{y}, \mathbf{x}),$$

for all $\mathbf{x}, \mathbf{y} \in V$.

Theorem

Every quadratic form $Q(\mathbf{x})$ on the space V can be represented in the form $Q(\mathbf{x}) = B(\mathbf{x}, \mathbf{x})$, where B is a symmetric bilinear form, and moreover, for the given quadratic form Q, the bilinear form B is unique.

Proof.

To construct such a bilinear form B from a given quadratic form, we can do the following,

$$B(\mathbf{x}, \mathbf{y}) = \frac{1}{4}\left(Q(\mathbf{x}+\mathbf{y}) - Q(\mathbf{x}-\mathbf{y})\right)$$

□

The following result is for antisymmetric forms.

Theorem

For every antisymmetric bilinear form $Q(\mathbf{x}, \mathbf{y})$ on the space V, we have

$$Q(\mathbf{x}, \mathbf{x}) = 0.$$

Conversely, if above equality is satisfied for every vector in V, then the bilinear form $Q(\mathbf{x}, \mathbf{y})$ is antisymmetric.

Reduction to Canonical Form

Let $Q(\mathbf{x}, \mathbf{y})$ be a symmetric bilinear form defined on a finite dimensional vector space V.

Definition

Vectors $\mathbf{x}$ and $\mathbf{y}$ are said to be orthogonal if $Q(\mathbf{x}, \mathbf{y}) = 0$.

The dimensionality and space decomposition property of V with respect to Q is the same as that for Euclidean inner product.

Theorem

For every quadratic form $\tilde{Q}(\mathbf{x})$, there exists a basis in which the form can be written as

$$\tilde{Q}(\mathbf{x}) = \lambda_1 x_1^2 + ... + \lambda_n x_n^2,$$

where $x_1, ..., x_n$ are the coordinates of the vector $\mathbf{x}$ in this basis.

Proof.

Let $Q(\mathbf{x}, \mathbf{y})$ be a symmetric bilinear form associated with the quadratic form $\tilde{Q}(\mathbf{x})$. If $\tilde{Q}(\mathbf{x})$ is identically zero, then the theorem is true. If the quadratic form $\tilde{Q}(\mathbf{x})$ is not identically zero, then there exists a vector $\mathbf{v}_1$ such that $\tilde{Q}(\mathbf{v}_1) = Q(\mathbf{v}_1, \mathbf{v}_1) \neq 0$. This implies that the restriction of the bilinear form Q to the subspace $V' = \langle \mathbf{v}_1 \rangle$ is nonsingular, and therefore, for the subspace $V' = \langle \mathbf{v}_1 \rangle$ we have the decomposition

$$V = \langle \mathbf{v}_1 \rangle \oplus \langle \mathbf{v}_1 \rangle^{\perp}.$$

Since $\dim \langle \mathbf{v}_1 \rangle = 1$, then we have that $\dim \langle \mathbf{v}_1 \rangle^{\perp} = n - 1$.
By induction, we may assume the theorem to have been proved for the space $\langle \mathbf{v}_1 \rangle^{\perp}$. Thus in this space there exists a basis $\mathbf{v}_2, \ldots, \mathbf{v}_n$ such that $Q(\mathbf{v}_i, \mathbf{v}_j) = 0$ for all $i \neq j \geq 2$. Then in the basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$, the quadratic form $\tilde{Q}(\mathbf{x})$ can be written as the form in the theorem. □

Applications: Quadratic Forms and Max/Min Problems

Definition

Let $X \subset \mathbb{R}^n$, and let $\mathbf{a} \in X$. The function $f : X \to \mathbb{R}$ has a *global maximum* at $\mathbf{a}$ if $f(\mathbf{x}) \leq f(\mathbf{a})$; the function f has a *local maximum* at $\mathbf{a}$ if, for some $\delta > 0$, we have $f(\mathbf{x}) \leq f(\mathbf{a})$ for all $\mathbf{x} \in B(\mathbf{a}, \delta) \cap X$. We say $\mathbf{a}$ is a local or global maximum point of f. The minimum can be defined analogously. If $\mathbf{a}$ is either a local maximum or local minimum point, we say it is an *extremum*.

Lemma

Suppose f is defined on some neighborhood of the extremum $\mathbf{a}$ and f is differentiable at $\mathbf{a}$. Then $Df(\mathbf{a}) = \mathbf{0}$, or, equivalently, $\nabla f(\mathbf{a}) = \mathbf{0}$.

Definition

Suppose f is differentiable at $\mathbf{a}$. We say $\mathbf{a}$ is a *critical point* if $Df(\mathbf{a}) = \mathbf{0}$. A critical point $\mathbf{a}$ with the property that $f(\mathbf{x}) < f(\mathbf{a})$ for some $\mathbf{x}$ near $\mathbf{a}$ and $f(\mathbf{x}) > f(\mathbf{a})$ for other $\mathbf{x}$ near $\mathbf{a}$ is called a *saddle point*.

Just as the second derivative test in single-variable calculus often allows us to differentiate between local minima and local maxima, there si something quite analogous in the multivariable case.

Lemma

Suppose $g : [0, 1] \to \mathbb{R}$ is twice differentiable. Then

$$g(1) = g(0) + g'(0) + \frac{1}{2}g''(\xi) \quad \textit{for some } 0 < \xi < 1.$$

The second derivative in the multivariable setting becomes a quadratic form.

Definition

Assume $f \in C^2$ in a neighborhood of $\mathbf{a}$. Define the symmetric matrix

$$\mathrm{Hess}(f)(\mathbf{a}) = \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a})\right].$$

Define the associated quadratic form $H_{f,\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}$ by

$$H_{f,\mathbf{a}}(\mathbf{h}) = \mathbf{h}^\top \left(\mathrm{Hess}(f)(\mathbf{a})\right) \mathbf{h} = \sum_{i,j}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a}) h_i h_j.$$

Proposition

Suppose $f : B(\mathbf{a}, r) \to \mathbb{R}$ *is* C^2. *Then for all* $\mathbf{h}$ *with* $\|h\| < r$ *we have*

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + Df(\mathbf{a})\mathbf{h} + \frac{1}{2}H_{f,\mathbf{a}+\xi\mathbf{h}}(\mathbf{h}) \quad \textit{for some } 0 < \xi < 1.$$

Consequently,

$$f(\mathbf{a}+\mathbf{h}) = f(\mathbf{a}) + Df(\mathbf{a})\mathbf{h} + \frac{1}{2}H_{f,\mathbf{a}}(\mathbf{h}) + \boldsymbol{\epsilon}(\mathbf{h})$$

where

$$\boldsymbol{\epsilon}(\mathbf{h}) / \|h\|^2 \to 0 \quad \textit{as } \mathbf{h} \to \mathbf{0}.$$

Proof

Using the chain rule,

$$g'(t) = Df(\mathbf{a} + t\mathbf{h})\mathbf{h} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\mathbf{a} + t\mathbf{h})h_i$$

$$g''(t) = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a} + t\mathbf{h})h_j \right) h_i = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{a} + t\mathbf{h})h_i h_j = H_{f,\mathbf{a}+t\mathbf{h}}.$$

Substitution yields the first result.

Proof. cont.

Since f is C^2, given any $\epsilon > 0$, there is $\delta > 0$ such that whenever $\mathbf{v} < \delta$ we have

$$\left\|\text{Hess}(f)(\mathbf{a}+\mathbf{v}) - \text{Hess}(f)(\mathbf{a})\right\| < \epsilon.$$

Using the Cauchy-Schwarz inequality, we have that

$$\left|\mathbf{h}^\top A\mathbf{h}\right| \leq \|A\| \, \|h\|^2.$$

So whenever $\|h\| < \delta$, we have, for any $0 < \xi < 1$,

$$|H_{f,\mathbf{a}+\xi\mathbf{h}}(\mathbf{h}) - H_{f,\mathbf{a}}\mathbf{h}| < \epsilon \|h\|^2.$$

By definition, $\boldsymbol{\epsilon}(\mathbf{h}) = \frac{1}{2}\left(H_{f,\mathbf{a}+\xi\mathbf{h}} - H_{f,\mathbf{a}}(\mathbf{h})\right)$, so

$$\frac{|\boldsymbol{\epsilon}(\mathbf{h})|}{\|\mathbf{h}\|^2} = \frac{|H_{f,\mathbf{a}+\xi\mathbf{h}}(\mathbf{h}) - H_{f,\mathbf{a}}(\mathbf{h})|}{2\|h\|^2} < \frac{\epsilon}{2}$$

whenever $\|\mathbf{h}\| < \delta$.

Definition

Given a symmetric $n \times n$ matrix A, we say the associated quadratic form $Q : \mathbb{R}^n \to \mathbb{R}$, $Q(\mathbf{x}) = \mathbf{x}^\top A\mathbf{x}$, is

- *positive definite* if $Q(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$,
- *negative definite* if $Q(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$,
- *positive semidefinite* if $Q(\mathbf{x}) \geq 0$ for all $\mathbf{x}$ and $= 0$ for some $\mathbf{x} \neq \mathbf{0}$,
- *negative semidefinite* if $Q(\mathbf{x}) \leq 0$ for all $\mathbf{x}$ and $= 0$ for some $\mathbf{x} \neq \mathbf{0}$,
- *indefinite* if $Q(\mathbf{x}) > 0$ for some $\mathbf{x}$ and $Q(\mathbf{x}) < 0$ for other $\mathbf{x}$.

Theorem

Suppose $f : B(\mathbf{a}, r) \to \mathbb{R}$ is C^2 and $\mathbf{a}$ is a critical point. If $H_{f,\mathbf{a}}$ is positive (resp., negative) definite, then $\mathbf{a}$ is a local minimum (resp., maximum) point; if $H_{f,\mathbf{a}}$ is indefinite, then $\mathbf{a}$ is a saddle point. If $H_{f,\mathbf{a}}$ is semidefinite, we can draw no conclusions.