Review on Linear Transformation, Orthogonality and Determinants I

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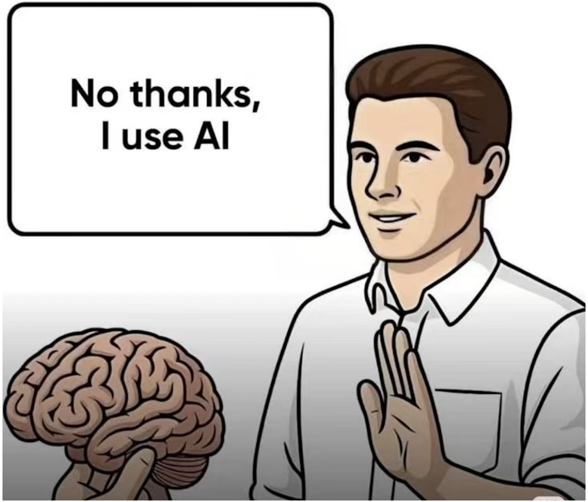
No thanks,
I use AI

Linear Transformation

Definition

A map $T : V \to W$ between two vector spaces is linear if

$$T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), \quad T(c\mathbf{v}) = cT(\mathbf{v})$$

Matrix Representation

Given a basis of V, say $B = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$, and a basis of W, say $C = \{\mathbf{w}_1, ..., \mathbf{w}_m\}$. For any $\mathbf{v} \in V$, it has a linear expression in terms of B:

$$\mathbf{v} = x_1\mathbf{v}_1 + ... + x_n\mathbf{v}_n$$

The Matrix $[T]_B^C$ representing T when choosing B and C as basis of V and W is defined:

$$[T]_B^C = \begin{bmatrix} [T(\mathbf{v}_1)]_C & [T(\mathbf{v}_2)]_C & \cdots & [T(\mathbf{v}_n)]_C \end{bmatrix}$$

Matrix Representation cont.

The meaning of the matrix $[T]_B^C$: the j-th column is the coordinate vector of $T(\mathbf{v}_j)$ in basis C.
Then for any vector $\mathbf{v} \in V$, the linear transformation T acting on $\mathbf{v}$ can be reduced to a Matrix-Vector multiplication:

$$[T(\mathbf{v})]_C = [T]_B^C \cdot \mathbf{x}$$

which means: the coordinate of $T(\mathbf{v})$ in basis C equals the matrix $[T]_B^C$ times $\mathbf{x}$, the coordinate of $\mathbf{v}$ in basis B.

Two perspectives of $\mathbb{R}^n$

- $\mathbb{R}^n$ is a vector space by itself;
- $\mathbb{R}^n$ is a coordinate space of any n-dimensional vector space V over $\mathbb{R}$, with a basis specified.

Change of basis on V (two different bases, same transformation)

Let V be a vector space with two different bases:

- $B = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$
- $B' = \{\mathbf{v}'_1, ..., \mathbf{v}'_n\}$

Definition

The change of basis matrix from B to B' is the matrix P (or $P_B^{B'}$) whose columns are the coordinates of the B'-basis vectors expressed in basis B:

$$P_B^{B'} = \begin{bmatrix} [\mathbf{v}'_1]_B & [\mathbf{v}'_2]_B & \cdots & [\mathbf{v}'_n]_B \end{bmatrix}$$

For any $\mathbf{v} = x_1\mathbf{v}_1 + ... + x_n\mathbf{v}_n = x'_1\mathbf{v}'_1 + ... + x_n\mathbf{v}'_n \in V$, the coordinates satisfy:

$$[\mathbf{v}]_{B'} = (P_B^{B'})^{-1}[\mathbf{v}]_B,$$

where $[\mathbf{v}]_B = (x_1, ..., x_n)^\top$ and $[\mathbf{v}]_{B'} = (x'_1, ..., x'_n)^\top$