

Eigenvalues and Eigenvectors

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Artist view of a Venus-Earth collision.(J. Vidal-Madjar),



Original link:

http://www.scholarpedia.org/article/Stability_of_the_solar_system

Example

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation defined by multiplying by

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

It is difficult to understand this mapping until we discover that if we take

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Then $T(\mathbf{v}_1) = 4\mathbf{v}_1$ and $T(\mathbf{v}_2) = \mathbf{v}_2$, so that the matrix for T with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ is the diagonal matrix

$$[T]_B = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

The Characteristic Polynomial

A linear transformation $T : V \rightarrow V$ is *diagonalizable* if there is an basis $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for V so that the matrix for T with respect to that basis is diagonal. This means precisely that, for some scalars $\lambda_1, \dots, \lambda_n$, we have

$$T(\mathbf{v}_1) = \lambda_1 \mathbf{v}_1$$

$$T(\mathbf{v}_2) = \lambda_2 \mathbf{v}_2$$

$$\vdots$$

$$T(\mathbf{v}_n) = \lambda_n \mathbf{v}_n.$$

An $n \times n$ matrix A is diagonalizable if the associated linear transformation $\mu_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is diagonalizable. So A is diagonalizable precisely when there is a basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for \mathbb{R}^n with the property that $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$. We can write these equations in matrix form:

$$A \begin{bmatrix} | & | & \dots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & \dots & | \end{bmatrix} = \begin{bmatrix} | & | & \dots & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \\ | & | & \dots & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Thus, if we let P be the $n \times n$ matrix whose columns are the vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ and Λ be the $n \times n$ diagonal matrix with diagonal entries $\lambda_1, \dots, \lambda_n$, then we have

$$AP = P\Lambda, \quad \text{and so} \quad P^{-1}AP = \Lambda.$$

Definition

Let $T : V \rightarrow V$ be a linear transformation. A *nonzero* vector $\mathbf{v} \in V$ is called an *eigenvector* of T if there is a scalar λ so that $T(\mathbf{v}) = \lambda\mathbf{v}$. The scalar λ is called the associated *eigenvalue* of T .

This definition leads to a convenient reformulation of diagonalizability:

Proposition

The linear transformation $T : V \rightarrow V$ is diagonalizable if and only if there is a basis for V consisting of eigenvectors of T .

An important question is how can we find eigenvectors. We most of the time will have a matrix representation for a linear map. Let's say A denote the matrix for T with respect to some basis. We start by observing that the set of eigenvectors with eigenvalue λ , together with the zero vector, forms a subspace.

Lemma

Let A be an $n \times n$ matrix, and let λ be any scalar. Then

$$\mathbf{E}(\lambda) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \lambda\mathbf{x}\} = \mathbf{N}(A - \lambda I)$$

is a subspace of \mathbb{R}^n . Moreover, $\mathbf{E}(\lambda) \neq \{\mathbf{0}\}$ if and only if λ is an eigenvalue, in which case we call $\mathbf{E}(\lambda)$ the λ -eigenspace of the matrix A .

Proof.

The null space $\mathbf{N}(A - \lambda I)$ is a subspace. By definition λ is an eigenvalue if and only if there is a nonzero vector in $\mathbf{E}(\lambda)$ □

The following result is a computational tool for finding eigenvalues.

Proposition

Let A be an $n \times n$ matrix. Then λ is an eigenvalue of A if and only if $\det(A - \lambda I) = 0$.

Proof.

From above lemma we know that λ is an eigenvalue if and only if the matrix $A - \lambda I$ is singular. The determinant of a singular matrix must be zero. □

Example

Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 1 \\ -3 & 7 \end{bmatrix}$$

Definition

Let A be a square matrix. Then $p(t) = p_A(t) = \det(A - tI)$ is called the *characteristic polynomial* of A .

Lemma

If A and B are similar matrices, the $p_A(t) = p_B(t)$.

Proof.

Suppose $B = P^{-1}AP$. Then

$$p_B(t) = \det(B - tI) = \det(P^{-1}AP - tI) = \det(P^{-1}(A - tI)P) = \det(A - tI) = p_A(t).$$

□

Diagonalizability

Theorem

Let $T : V \rightarrow V$ be a linear transformation. Suppose $\mathbf{v}_1, \dots, \mathbf{v}_k$ are eigenvectors of T with distinct corresponding eigenvalues $\lambda_1, \dots, \lambda_k$. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ is a linearly independent set of vectors.

Proof.

Let m be the largest number between 1 and k so that $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is linear independent. We need to show that $m = k$.

By contradiction, suppose $m < k$. Then $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is linearly independent and $\{\mathbf{v}_1, \dots, \mathbf{v}_m, \mathbf{v}_{m+1}\}$ is linearly dependent. Then we have

$$\mathbf{v}_{m+1} = c_1 \mathbf{v}_1 + \dots + c_m \mathbf{v}_m$$

for some c_i 's. Then using $T\mathbf{v}_i = \lambda_i \mathbf{v}_i$, we can have

$$\mathbf{0} = (T - \lambda_{m+1}I)\mathbf{v}_{m+1} = (T - \lambda_{m+1}I)(c_1 \mathbf{v}_1 + \dots + c_m \mathbf{v}_m) \quad (1)$$

$$= c_1(\lambda_1 - \lambda_{m+1})\mathbf{v}_1 + \dots + c_m(\lambda_m - \lambda_{m+1})\mathbf{v}_m. \quad (2)$$

Since $\lambda_i - \lambda_{m+1} \neq 0$ for all i , and since $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ is linearly independent, it has to be that

$$c_1 = c_2 = \dots = c_m = 0,$$

contradicting $\mathbf{v}_{m+1} \neq \mathbf{0}$. Thus, $m < k$ is impossible.

We now arrive at our first result that gives a sufficient condition for a linear transformation to be diagonalizable. Note that we require the eigenvalues to be real numbers, the situation with complex eigenvalues will be discussed later.

Corollary

Suppose V is an n -dimensional vector space and $T : V \rightarrow V$ has n distinct (real) eigenvalues. Then T is diagonalizable.

Proof.

Consider the matrix of T with respect to the basis of eigenvectors. □

The previous condition is “sufficient”. There are many diagonalizable matrices with repeated eigenvalues. We discuss two ways in which the hypotheses of Corollary can fail: The characteristic polynomial may have complex roots or it may have repeated roots.

Example

Consider the matrix

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

The characteristic polynomial is

$$p(t) = t^2 - (\operatorname{tr}A)t + \det A = t^2 - \sqrt{2}t + 1,$$

whose roots are

$$\lambda = \frac{1+i}{\sqrt{2}} \quad \text{and} \quad \frac{1-i}{\sqrt{2}}.$$

The geometric meaning of A is to rotate the plane through an angle of $\frac{\pi}{4}$. Thus, it comes as no surprise that A has no real eigenvectors, as there can be no line through the origin that is unchanged after a rotation.

Example

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$$

Its characteristic polynomial is $p(t) = t^2 - 4t + 4$, so 2 is a repeated eigenvalue. Now let's find the corresponding eigenvectors:

$$\mathbf{N}(A - 2I) = \mathbf{N} \left(\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right) = \mathbf{N} \left(\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \right)$$

is one-dimensional, with basis $\{[1, 1]^T\}$. A cannot be diagonalized since this is the only eigenvector and there cannot be basis of eigenvectors.

An “iff” for diagonalizability

Definition

Let λ be an eigenvalue of a linear transformation. The *algebraic multiplicity* of λ is its multiplicity as a root of the characteristic polynomial $p(t)$, i.e., the highest power of $t - \lambda$ dividing $p(t)$. The *geometric multiplicity* of λ is the dimension of the λ -eigenspace $\mathbf{E}(\lambda)$.

Proposition

Let λ be an eigenvalue of algebraic multiplicity m and geometric multiplicity d . Then $1 \leq d \leq m$.

Proof.

Suppose λ is an eigenvalue of the linear transformation T . Then $d = \dim \mathbf{E}(\lambda) \geq 1$. Now choose a basis $\mathbf{v}_1, \dots, \mathbf{v}_d$ for $\mathbf{E}(\lambda)$ and extend it to a basis $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ for V (how?). Then the matrix for T with respect to the basis \mathcal{B} is

$$A = \begin{bmatrix} \lambda I_d & B \\ 0 & C \end{bmatrix}.$$

Then the determinant of $A - tI$ satisfies

$$\det(A - tI) = \det((\lambda - t)I_d) \det(C - tI) = (\lambda - t)^d \det(C - tI).$$

Since the characteristic polynomial does not depend on the choice of basis, and since $(t - \lambda)^m$ is the largest power of $(t - \lambda)$ dividing the characteristic polynomial, it follows that $d \leq m$. □

Theorem

Let $T : V \rightarrow V$ be a linear transformation. Let its distinct eigenvalues be $\lambda_1, \dots, \lambda_k$ and assume these are all real numbers. Then T is diagonalizable if and only if the geometric multiplicity, d_i , of each λ_i equals its algebraic multiplicity, m_i .

Proof

Let V be an n -dimensional vector space. Then the characteristic polynomial of T has degree n , and we have

$$p(t) = \pm(t - \lambda_1)^{m_1}(t - \lambda_2)^{m_2} \cdots (t - \lambda_k)^{m_k};$$

therefore,

$$n = \sum_{i=1}^k m_i.$$

Proof cont.

Suppose T is diagonalizable. Then there exists a basis B consisting of eigenvectors. At most d_i of these basis vectors lie in $\mathbf{E}(\lambda_i)$, and so

$$n \leq \sum_{i=1}^k d_i.$$

On the other hand, by previous proposition, we know that $d_i \leq m_i$ for all i . Then we have

$$n \leq \sum_{i=1}^k d_i \leq \sum_{i=1}^k m_i = n,$$

which implies $d_i = m_i$ for all $i = 1, \dots, k$.

Proof cont.

Conversely, suppose $d_i = m_i$ for $i = 1, \dots, k$. If we choose a basis B_i for each eigenspace $\mathbf{E}(\lambda_i)$ and let $B = B_1 \cup \dots \cup B_k$, then we need to verify B is a basis for V . Since there are n vectors in B , so we need only check that the set of vectors is linearly independent. This can be done through the following exercise.

Ex.

- ▶ Let λ and μ be distinct eigenvalues of a linear transformation. Suppose $\{\mathbf{v}_1, \dots, \mathbf{v}_k\} \subset \mathbf{E}(\lambda)$ is linearly independent and $\{\mathbf{w}_1, \dots, \mathbf{w}_l\} \subset \mathbf{E}(\mu)$ is linearly independent. Prove that $\{\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{w}_1, \dots, \mathbf{w}_l\}$ is linearly independent.
- ▶ More generally, if $\lambda_1, \dots, \lambda_k$ are distinct and $\{\mathbf{v}_1^{(i)}, \dots, \mathbf{v}_{d_i}^{(i)}\} \subset \mathbf{E}(\lambda_i)$ is linearly independent for $i = 1, \dots, k$, prove that $\{\mathbf{v}_j^{(i)} : i = 1, \dots, k, j = 1, \dots, d_i\}$ is linearly independent.

Spectral Theorem

We now turn to the study of a large class of diagonalizable matrices, the symmetric matrices. Recall that a square matrix A is symmetric when $A = A^T$. Start with a general symmetric 2 matrix

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

whose characteristic polynomial is $p(t) = t^2 - (a + c)t + (ac - b^2)$. By the quadratic formula, its eigenvalues are

$$\lambda = \frac{(a + c) \pm \sqrt{(a - c)^2 + 4b^2}}{2}.$$

The first thing we notice is that both eigenvalues are real. Moreover, the corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} b \\ \lambda_1 - a \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} \lambda_2 - c \\ b \end{bmatrix}$$

note that

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = b(\lambda_2 - c) + (\lambda_1 - a)b = 0,$$

so the eigenvectors are orthogonal. Since there is an orthogonal basis for \mathbb{R}^2 consisting of eigenvectors of A , we have an orthonormal basis consisting of eigenvectors of A . That is, by an appropriate rotation of the usual basis, we obtain a diagonalizing basis for A .

In general, we have the following important result.

Theorem

Let A be a symmetric $n \times n$ matrix. Then

- 1. The eigenvalues of A are real.*
- 2. There is an orthonormal basis for \mathbb{R}^n consisting of eigenvectors of A . That is, there is an orthogonal matrix Q so that $Q^{-1}AQ = \Lambda$ is diagonal.*

Proof.

1. Let $\lambda = a + bi$ be an eigenvalue of A , and consider the real matrix

$$S = (A - (a + bi)I)(A - (a - bi)I) = (A - aI)^2 + b^2I.$$

Since

$$\det(A - \lambda I) = 0,$$

it follows that $\det S = 0$. Thus S is singular, and then there is a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ such that $S\mathbf{x} = \mathbf{0}$.

Since $S\mathbf{x} = \mathbf{0}$, then the dot product $S\mathbf{x} \cdot \mathbf{x} = 0$, which gives

$$0 = S\mathbf{x} \cdot \mathbf{x} = \|(A - aI)\mathbf{x}\|^2 + b^2 \|\mathbf{x}\|^2.$$

Then it must be true that $(A - aI)\mathbf{x} = \mathbf{0}$, and $b = 0$. Thus $\lambda = a$ is a real number.

proof. cont.

2. Let λ_1 be one of the eigenvalues of A , and choose a unit vector \mathbf{q}_1 that is an eigenvector with eigenvalue λ_1 . Choose $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ to be any orthonormal basis for $((\mathbf{q}_1))^\perp$. Then the matrix for the linear transformation with respect to the basis $\{\mathbf{q}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is

$$B = \begin{bmatrix} \lambda_1 & * & \dots & * \\ 0 & & & \\ \vdots & & C & \\ 0 & & & \end{bmatrix}$$

for some $(n - 1) \times (n - 1)$ matrix C and some entries $*$.

proof. cont.

By the change-of-basis formula, we have

$$B = Q^{-1}AQ = Q^{\top}AQ,$$

since Q is an orthogonal matrix. Therefore,

$$B^{\top} = (Q^{\top}AQ)^{\top} = Q^{\top}A^{\top}Q = Q^{\top}AQ = B.$$

Since B is symmetric, we have that the entries * are all 0 and that C is symmetric. This process can be continued and we will end up with a orthonormal basis consisting of eigenvectors of A .