Determinant

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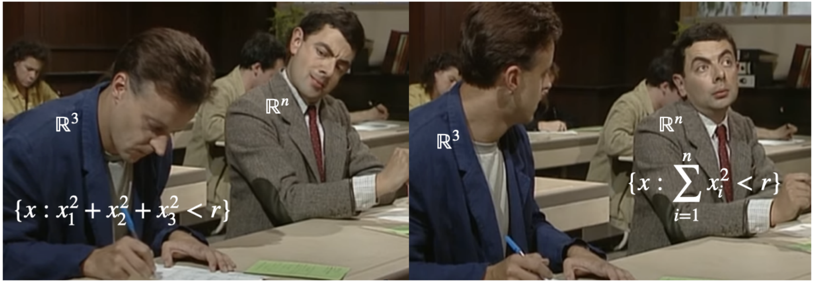
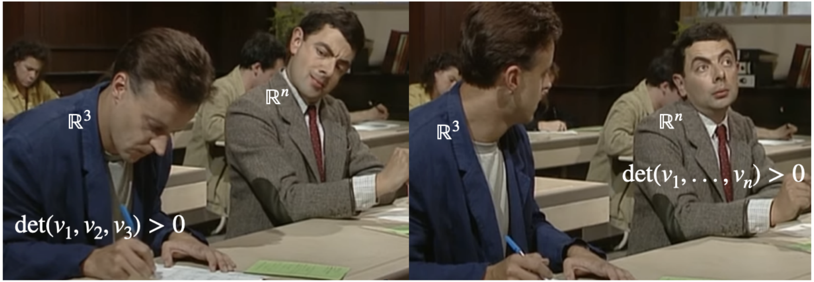
The Art of Generalization

A Ball



Orientation



Determinant as Area Scaling Factor (2D)

Definition (Determinant of 2×2 Matrix)

For $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the **determinant** is:

$$\det(A) = ad - bc$$

Proposition (Geometric Interpretation)

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be linear with standard matrix A. For a parallelogram P with area $Area(P)$:

$$Area(T(P)) = |\det(A)| \cdot Area(P)$$

- $|\det(A)|$: Scaling factor for area - Sign of $\det(A)$: Orientation (preserved if $+$, reversed if $-$)

Determinant as Volume Scaling Factor (3D)

Definition (Determinant of 3×3 Matrix)

For $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$:

$$\det(A) = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Proposition (3D Geometric Interpretation)

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be linear with standard matrix A. For a parallelepiped Q with volume $Vol(Q)$:

$$Vol(T(Q)) = |\det(A)| \cdot Vol(Q)$$

- Sign: Orientation (right-hand vs. left-hand rule)

Key Observations (2D/3D Determinants)

For $A \in \mathbb{R}^{n \times n}$ $(n = 2, 3)$:

1. $\det(I_n) = 1$ (identity matrix scales area/volume by 1)
2. A is invertible $\iff$ $\det(A) \neq 0$ (non-degenerate transformation)
3. If A is singular (columns linearly dependent):

$$\det(A) = 0 \quad \text{(collapsed to line/plane, area/volume} = 0\text{)}$$

4. For rotation/scaling matrices: - Rotation $(2D)$: $\det(A) = 1$ (area preserved, orientation same) - Scaling $(2D)$: $A = \begin{bmatrix} k & 0 \\ 0 & l \end{bmatrix}$, $\det(A) = kl$ (area scales by kl)

Example

$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$: $\det(A) = 4 - 1 = 3$ (area scales by 3, orientation preserved)

Fundamental Properties (Axiomatic Definition)

The **determinant function** $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is the unique function satisfying:

1. **Alternating (columns)**: Swapping two columns flips the sign:
$$\det([v_1 \dots v_j \dots v_k \dots v_n]) = -\det([v_1 \dots v_k \dots v_j \dots v_n])$$
2. **Scaling**: For $A = [v_1 \dots v_j \dots v_n]$:
$$\det([v_1 \dots cv_j \dots v_n]) = c\det([v_1 \dots v_j \dots v_n])$$
3. **Volume invariance**: For $A' = [v_1 \dots v_j + cv_k \dots v_n]$:
$$\det(A) = \det(A')$$
4. **Normalization**: $\det(I_n) = 1$

Corollary

If two columns of A are identical: $\det(A) = 0$ (swap columns $\to$ $\det(A) = -\det(A)$).

Derived Property 1: Scalar Multiplication on Columns

Proposition

Multiplying a single column of A by a scalar c scales the determinant by c:

$$\det([v_1 \dots cv_j \dots v_n]) = c \cdot \det([v_1 \dots v_j \dots v_n])$$

Proof.

Use multilinearity with $d = 0$:

$$\det([v_1 \dots cv_j + 0 \cdot w_j \dots v_n]) = c \det([v_1 \dots v_j \dots v_n]) + 0 = c \det(A)$$

□

Corollary

For $A \in \mathbb{R}^{n\times n}$ and scalar c:

$$\det(cA) = c^n \det(A) \quad \text{(all } n \text{ columns scaled by } c\text{)}$$

Example

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $2A = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$: $\det(A) = -2$, $\det(2A) = 16 - 24 = -8 = 2^2(-2)$

Derived Property 2: Elementary Column Operations

Let $A \in \mathbb{R}^{n\times n}$:

1. **Swap columns**: $A' = \text{swap columns } j, k \implies \det(A') = -\det(A)$
2. **Scale column**: $A' = \text{scale column } j \text{ by } c \implies \det(A') = c\det(A)$
3. **Add multiple of column to another**:
   $A' = \text{column } j + c \cdot \text{column } k \implies \det(A') = \det(A)$

Example

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A' = \begin{bmatrix} 1 & 2+3(1) \\ 3 & 4+3(3) \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 3 & 13 \end{bmatrix}$: $\det(A) = -2$,
$\det(A') = 13 - 15 = -2 = \det(A)$

Key Use Case

Reduce A to upper triangular form with column operations (preserves determinant sign/magnitude).

Derived Property 3: Invertibility and Singularity

Theorem

A matrix $A \in \mathbb{R}^{n\times n}$ is *invertible* if and only if $\det(A) \neq 0$.

Sketch.

- If A is invertible: A is product of elementary matrices. $\det(A) = \prod \det(E_i) \neq 0$ (elementary matrices have non-zero determinant). - If A is singular: Columns are linearly dependent. Write column $j = \sum_{k\neq j} c_k v_k$. Use multilinearity/alternating property to show $\det(A) = 0$. □

Corollary

Columns of A are linearly independent $\iff \det(A) \neq 0$.

Example

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$: $\det(A) = 4 - 4 = 0$ (columns dependent, singular).

Derived Property 4: Determinant of a Product

Theorem (Product Rule)

For $A, B \in \mathbb{R}^{n\times n}$:

$$\det(AB) = \det(A)\det(B)$$

Geometric Intuition.

Let $T_A(x) = Ax$, $T_B(x) = Bx$. The composition $T_A \circ T_B(x) = ABx$ scales volume by:

$$|\det(AB)| = |\det(A)| \cdot |\det(B)|$$

Sign matches product of signs (orientation preserved/reversed twice). □

Corollary

For invertible A: $\det(A^{-1}) = \frac{1}{\det(A)}$ (since $AA^{-1} = I \implies \det(A)\det(A^{-1}) = 1$).

Example

$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$: $\det(A) = 6$, $\det(B) = -1$,

$\det(AB) = \det(\begin{bmatrix} 0 & 2 \\ 3 & 0 \end{bmatrix}) = -6 = 6(-1)$.

Derived Property 5: Determinant of Transpose

Theorem

For any $A \in \mathbb{R}^{n \times n}$:

$$\det(A^T) = \det(A)$$

Key Idea.

- Elementary row operations affect determinant the same way as column operations (by transpose symmetry). - Reduce A to upper triangular form U (row ops): $\det(A) = \pm\det(U)$. - A^T reduces to lower triangular form U^T (column ops): $\det(A^T) = \pm\det(U^T) = \pm\det(U) = \det(A)$. □

Corollary

All determinant properties for columns apply to rows (e.g., swapping rows flips sign).

Example

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$: $\det(A) = -2$, $\det(A^T) = 4 - 6 = -2$.

Minors and Cofactors

Definition (Minor)

For $A \in \mathbb{R}^{n\times n}$, the **minor** M_{ij} is the determinant of the $(n-1)\times(n-1)$ matrix obtained by deleting row i and column j of A.

Definition (Cofactor)

The **cofactor** C_{ij} is:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

(Sign pattern: checkerboard $\begin{bmatrix} + & - & + & \dots \\ - & + & - & \dots \\ + & - & + & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$)

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \text{: - } M_{12} = \det(\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}) = 36 - 42 = -6 \text{ - } C_{12} = (-1)^{1+2}(-6) = 6$$

Cofactor Expansion (Laplace Expansion)

Theorem (Cofactor Expansion)

For any $A \in \mathbb{R}^{n \times n}$:

1. Expand along row i:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in} = \sum_{j=1}^{n} a_{ij}C_{ij}$$

2. Expand along column j:

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} = \sum_{i=1}^{n} a_{ij}C_{ij}$$

Key Tip

Expand along rows/columns with the most zeros to simplify computation!

Example

$A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 5 & 6 \\ 0 & 8 & 9 \end{bmatrix}$: Expand along row 1 (has zero):

$\det(A) = 1 \cdot C_{11} + 0 \cdot C_{12} + 3 \cdot C_{13} = 1(45 - 48) + 3(32 - 0) = -3 + 96 = 93$

Example: Cofactor Expansion (4x4 Matrix)

Example

Compute $\det(A)$ for $A = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 \\ 3 & 0 & 2 & 0 \\ 4 & 5 & 6 & 7 \end{bmatrix}$ (expand along row 1):

Step 1: Compute cofactors for row 1

- $C_{11} = (-1)^2 \det(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 5 & 6 & 7 \end{bmatrix}) = 1(-1)(14-0) = -14$ - $C_{12} = C_{13} = 0$ (zero entries in row 1) - $C_{14} = (-1)^5 \det(\begin{bmatrix} 1 & -1 & 0 \\ 3 & 0 & 2 \\ 4 & 5 & 6 \end{bmatrix}) = -1(1(0-10) + 1(18-8) + 0) = -1(-10+10) = 0$

Step 2: Expand along row 1

$$\det(A) = 2(-14) + 0 + 0 + 1(0) = -28$$

Computing Determinants via Row Reduction

Steps to Compute $\det(A)$ with Row Reduction: 1. Reduce A to upper triangular form U using elementary row operations: - Swap rows: multiply determinant by -1 - Scale row: multiply determinant by scaling factor - Add multiple of row to another: determinant unchanged 2. $\det(U) = \prod_{i=1}^{n} u_{ii}$ (product of diagonal entries) 3. Adjust for row swaps/scalings to get $\det(A)$

Example

$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 3 & 7 & 10 \end{bmatrix}$: - $R2 \leftarrow R2 - 2R1$, $R3 \leftarrow R3 - 3R1$: $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (det unchanged) - $R3 \leftarrow R3 - R2$: $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (det unchanged) - $\det(A) = 1 \cdot 1 \cdot 0 = 0$ (singular matrix)

Adjugate Matrix and Inverse Formula

Definition (Adjugate Matrix)

The **adjugate** (or adjoint) of $A \in \mathbb{R}^{n\times n}$ is:

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

(Transpose of the cofactor matrix.)

Theorem (Inverse Formula)

For invertible A:

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

Example

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$: $\det(A) = -2$, $\operatorname{adj}(A) = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

Example: Inverse via Adjugate (3x3 Matrix)

Example

Find A^{-1} for $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$:

Step 1: Compute $\det(A)$

Expand along row 1: $\det(A) = 1(0-6) - 0 + 2(0-1) = -6 - 2 = -8$

Step 2: Compute cofactor matrix

$$C = \begin{bmatrix} -6 & 3 & -1 \\ 4 & -2 & 2 \\ -2 & 3 & 1 \end{bmatrix}$$

Step 3: Adjugate and inverse

$$\text{adj}(A) = C^T = \begin{bmatrix} -6 & 4 & -2 \\ 3 & -2 & 3 \\ -1 & 2 & 1 \end{bmatrix}$$

Cramer's Rule

Theorem (Cramer's Rule)

Let $Ax = b$ be a linear system with invertible $A \in \mathbb{R}^{n \times n}$. The unique solution is:

$$x_j = \frac{\det(A_j)}{\det(A)} \quad (j = 1, 2, \ldots, n)$$

where A_j is A with column j replaced by b.

Use Case

Useful for solving small systems ($n = 2, 3$) or finding a single component of x (avoids full matrix inversion).

Example

Solve $\begin{cases} x + 2y = 5 \\ 3x + 4y = 11 \end{cases}$: - $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\det(A) = -2$ - $A_1 = \begin{bmatrix} 5 & 2 \\ 11 & 4 \end{bmatrix}$, $\det(A_1) = 20 - 22 = -2$ - $A_2 = \begin{bmatrix} 1 & 5 \\ 3 & 11 \end{bmatrix}$, $\det(A_2) = 11 - 15 = -4$ - $x = -2/-2 = 1$, $y = -4/-2 = 2$

Example: Cramer's Rule (3x3 System)

Example

Solve $Ax = b$ where $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$:

Step 1: Compute $\det(A)$

$\det(A) = -8$ (from earlier example)

Step 2: Compute A_j and $\det(A_j)$

- $A_1 = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 3 \\ 1 & 2 & 0 \end{bmatrix}$, $\det(A_1) = 1(0-6) - 0 + 2(2-1) = -6 + 2 = -4$ - $A_2 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 1 & 1 & 0 \end{bmatrix}$, $\det(A_2) = 1(0-3) - 1(0-3) + 2(0-1) = -3 + 3 - 2 = -2$ - $A_3 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, $\det(A_3) = 1(1-2) - 0 + 1(0-1) = -1 - 1 = -2$

Step 3: Apply Cramer's Rule

$x_1 = -4/-8 = 1/2$, $x_2 = -2/-8 = 1/4$, $x_3 = -2/-8 = 1/4$

Determinant of Block Matrices (Optional)

For block matrices with square diagonal blocks:

1. Diagonal blocks:

$$A = \begin{bmatrix} B & 0 \\ 0 & C \end{bmatrix} \implies \det(A) = \det(B)\det(C)$$

   (Generalizes to k diagonal blocks: $\det(A) = \prod_{i=1}^{k} \det(B_i)$)

2. Triangular blocks:

$$A = \begin{bmatrix} B & D \\ 0 & C \end{bmatrix} \implies \det(A) = \det(B)\det(C)$$

Example

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \\ 0 & 0 & 5 & 6 \\ 0 & 0 & 7 & 8 \end{bmatrix} : \ \det(A) = \det(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}) \det(\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}) = (-2)(-2) = 4$$

Determinant of a Linear Transformation

Definition (Determinant of $T : V \to V$)

Let V be finite-dimensional ($\dim V = n$), B a basis for V, and $T : V \to V$ linear. The **determinant of T** is:

$$\det(T) = \det([T]_B)$$

Proposition (Invariance Under Change of Basis)

$\det(T)$ is *basis-independent*: For any bases B, B':

$$\det([T]_B) = \det([T]_{B'})$$

Proof.

$[T]_{B'} = P^{-1}[T]_B P \implies \det([T]_{B'}) = \det(P^{-1}) \det([T]_B) \det(P) = \det([T]_B)$ (since $\det(P^{-1}) \det(P) = 1$).

Example

$T : \mathbb{R}^2 \to \mathbb{R}^2$ (rotation by θ): $\det(T) = \det(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}) = 1.$

Determinant and Orientation

Definition (Orientation of a Basis)

A basis $B = \{v_1, \ldots, v_n\}$ for $\mathbb{R}^n$ is: - *Right-handed* (positive orientation): $\det([v_1 \ldots v_n]) > 0$ - *Left-handed* (negative orientation): $\det([v_1 \ldots v_n]) < 0$

Proposition (Orientation Preservation)

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear and invertible: - If $\det(T) > 0$: T preserves orientation (right-handed $\to$ right-handed) - If $\det(T) < 0$: T reverses orientation (right-handed $\to$ left-handed)

Determinant and Eigenvalues (Preview)

Definition (Eigenvalue/Eigenvector)

For $T : V \to V$, a scalar λ is an *eigenvalue* if there exists $v \neq 0$ (eigenvector) such that:

$$T(v) = \lambda v$$

Proposition

For $A \in \mathbb{R}^{n \times n}$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$:

$$\det(A) = \lambda_1 \lambda_2 \ldots \lambda_n$$

Example

$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ (eigenvalues 2,3): $\det(A) = 6 = 2 \cdot 3$ $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (eigenvalues $i, -i$): $\det(A) = 1 = i(-i)$

Key Takeaway

Determinant is product of scaling factors along eigenvector directions (geometric interpretation).