# Kazhdan's Property (T) and Structure theorem for locally symmetric spaces

Xiao Wang

**Abstract**

In this article, we discuss the structure at infinity of locally symmetric space $\Gamma \setminus G/K$. Where $G$ is a semisimple Lie group that satisfies Kazhdan's property (T), K is the maximal compact subgroup of $G$ and $\Gamma$ is a torsion-free discrete subgroup of $G$. Kazhdan's property (T) and semisimpleness of Lie group $G$ play essential roles in obtaining spectrum information of $\Gamma \setminus G/K$, and furthermore, in concluding number of ends of $\Gamma \setminus G/K$.

## 1 Introduction

This paper studies the structure at infinity of $\Gamma \setminus G/K$, where $G$ is a semisimple Lie group satisfying Kazhdan's property(T), K is maximal compact subgroup of $G$, $\Gamma$ is a discrete, torsion-free subgroup of $G$. Kevin Corlette announced in 1995 Fall Eastern Sectional Meeting at Boston that locally symmetric space $\Gamma \setminus G/K$ satisftying above conditions has finite many ends, at most one of which is of infinite volume. Our main result is as follows:

**Theorem 1.1.** *Let $G$ be a semisimple Lie group that satisfies Kazhdan's property (T), $K$ be the maximal compact subgroup of $G$ such that $G/K$ is a symmetric space without any compact factor and without any factor isometric to a real or complex hyperbolic space, $\Gamma$ be a discrete, torsion-free subgroup of $G$. Then $\Gamma \setminus G/K$ has at most one end with infinite volume, and the compliment of the end with infinite volume has finite volume.*

In this article, by an *end* we mean a connected component of the complement of some compact set which is not relatively compact.

This work is inspired by the paper of G. Carron and Emmanuel Pedon [4]. They point out that the Kazhdan's property(T) of $G$ and the assumption that $\Gamma \setminus G/K$ has infinite volume implies that the first eigenvalue of Laplacian on $\Gamma \setminus G/K$ is bounded away from 0. On the other hand, the fact that the space of $L^2$-harmonic 1-forms of $\Gamma \setminus G/K$ is trivial is a consequence of Matsushima formula, i.e. $\dim\mathcal{H}^1_{L^2}(\Gamma \setminus G/K) = 0$, see [15]. The number of infinite volume ends is bounded above by $\dim\mathcal{H}^1_{L^2}(\Gamma \setminus G/K) + 1$ according to [11] .

The result of this article is true for the space $G/K$ of any rank. For rank one case, since $Sp(n, 1)$ and $F_{4(-20)}$ has property (T), $G/K$ is one of the hyperbolic spaces $H^n_{\mathbb{H}}$ or $H^n_{\mathbb{O}}$, the quotient $\Gamma \setminus G/K$ is called exotic hyperbolic manifold. The domain of discontinuity under action $\Gamma$ is well defined for exotic hyperbolic spaces, an analogue of Burns' Theorem in complex geometry can be proved. Recall Burns' Theorem as follows:

**Theorem 1.2.** *Let $\Gamma$ be a torsion-free group of automorphisms of the unit ball $B$ in $\mathbb{C}^n$ with $n \geq 3$ and let $M = \Gamma \setminus B$. Assume that the limit set $\Lambda$ is a proper subset of $\partial B$ and that the quotient $\Gamma \setminus (\partial B \setminus \Lambda)$ has a compact component $A$. Then $M$ has only finitely many ends; all of which except for the end corresponding to $A$, are cusps. In fact, $M$ is diffeomorphic to a compact manifold with boundary.*

For quaternion and Cayley number hyperbolic manifolds, the result is presented:

**Theorem 1.3.** *Let $H^n_{\mathbb{K}}$ be a hyperbolic space, where $\mathbb{K} = \mathbb{H}$, $n \geq 2$ or $\mathbb{O}$, $n = 2$, $\Gamma$ be a discrete, torsion-free group of isometries of $H^n_{\mathbb{K}}$. Assume that the domain of discontinuity is not empty, and the quotient of discontinuity by the group action of $\Gamma$ has a compact connected component. Then $\Gamma \setminus H^n_{\mathbb{K}}$ has at most one end of infinite volume and finitely many ends in total.*

*Remark* 1.4. Kevin Corlette claimed finiteness of the number of ends in the higher rank case, but the author is not able to prove it while writing this article.

Our article is organized as follows. Basic homogeneous model for hyperbolic manifolds are given in section 2, together with some claims of notations which will be used throughout this paper. In section 3, we give basic facts about Kazhdan's property (T) (according to [1]). In section 4, results from P. Pansu [15] are recalled which implies that the first $L^2$ cohomology of $\Gamma \setminus G/K$ with coefficients in the unitary representation $\rho$ vanishes. In particular, $H^1(G, \rho) = 0$. In section 5, we collect necessary ingredients of harmonic function theory on smooth manifolds to complete the proof of Theorem 1.1.

*Acknowledgements* This paper is part of thesis under supervision of Dr. Ramachandran, The author would like to thank Dr. Ramachandran for all the help during the completion of this paper. The author also would like to thank G. Carron, P. Pansu and Luen Fai Tam for the helpful discussions.

# 2 Notations and Background

## 2.1 Homogeneous Models for Hyperbolic Manifolds

For $n \geq 2$ and $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or for $n = 2$, $\mathbb{K} = \mathbb{O}$. Let $H^n_{\mathbb{K}}$ be the Riemannian hyperbolic space of dimension $n$ over $\mathbb{K}$. Then$H^n_{\mathbb{K}}$ can be expressed as $G/K$, non-compact type space of rank one, where $G$ is a connected non-compact semisimple real Lie group with finite center and $K$ is a maximal compact subgroup of $G$ which consists of elements fixed by a Cartan involution $\theta$. They are listed as follows:

Table 1

| $\mathbb{K}$ | $G$ | $K$ |
|---|---|---|
| $\mathbb{R}$ | $SO(n,1)$ | $SO(n)$ |
| $\mathbb{C}$ | $SU(n,1)$ | $SU(n) \times U(1)$ |
| $\mathbb{H}$ | $Sp(n,1)$ | $Sp(n) \times Sp(1)$ |
| $\mathbb{O}$ | $F_{4(-20)}$ | $Spin(9)$ |

Let $\Gamma$ be any torsion-free discrete subgroup of $G$, so that the quotient $\Gamma \setminus G/K$ is a hyperbolic manifold, i.e. a complete Riemannian locally symmetric space with strictly negative curvature. Throughout this paper, we focus on $\Gamma \setminus H^n_{\mathbb{H}}$ and $\Gamma \setminus H^2_{\mathbb{O}}$.

# 3 Kazhdan's Property (T)

In this section, we collect some basic facts about Kazhdan's Property (T).

### 3.1 Unitary representation

Kazhdan's Property (T) for a topological group is defined in terms of unitary representations in Hilbert spaces. In this paper, Hilbert spaces are always complex. The inner product of two vectors $\xi$, $\eta$ in Hilbert space $\mathcal{H}$ is $\langle \xi, \eta \rangle$.

The *unitary group* $\mathcal{U}(\mathcal{H})$ of $\mathcal{H}$ is the group of all invertible bounded linear operators $U{:}\mathcal{H} \to \mathcal{H}$ which are unitary, namely such that, for all $\xi, \eta \in \mathcal{H}$,

$$\langle U\xi, U\eta \rangle = \langle \xi, \eta \rangle,$$

or equivalently such that $U^*U = UU^* = I$, where $U^*$ denotes the *adjoint* of $U$ and $I$ the identity operator on $\mathcal{H}$.

Let $G$ be a topological group. A *unitary representation* of G in $\mathcal{H}$ is a group homomorphism $\rho : G \to \mathcal{U}(\mathcal{H})$ which is *strongly continuous*, that is , such that the mapping

$$G \to \mathcal{H}, \quad g \mapsto \rho(g)\xi$$

is continuous for every $\xi \in \mathcal{H}$. We often write $(\rho, \mathcal{H})$ for such a representation.

**Definition 3.1.** Let $(\rho, \mathcal{H})$ be a unitary representation of a topological group$G$.

1. For a subset $Q$ of $G$ and real number $\epsilon > 0$, a vector $\xi \in \mathcal{H}$ is $(Q, \epsilon)$-*invariant* if

$$\sup_{x \in Q} ||\rho(x)\xi - \xi|| < \epsilon ||\xi||$$

2. The representation $(\rho, \mathcal{H})$ *almost has invariant vectors* if it has $(Q, \epsilon)$-invariant vectors for every *compact* subset $Q$ of $G$ and every $\epsilon > 0$.

3. The representation $(\rho, \mathcal{H})$ has non-zero *invariant vectors* if there exists $\xi \neq 0$ in $\mathcal{H}$ such that $\rho(g)\xi = \xi$ for all $g \in G$

**Definition 3.2.** Let $G$ be a topological group. A subset $Q$ of $G$ is a *Kazhdan set* if there exists $\epsilon > 0$ with the following property: every unitary representation $(\rho, \mathcal{H})$ of $G$ which has a $(Q, \epsilon)$-invariant vector also has a non-zero invariant vector.

In this case, $\epsilon > 0$ is called *Kazhdan constant* for $G$ and $Q$, and $(Q, \epsilon)$ is called a *Kazhdan pair* for $G$.

The group $G$ has *Kazhdan's Property (T)*, or is a *Kazhdan group*, if $G$ has a *compact* Kazhdan set.

In other words, $G$ has Kazhdan's Property (T) if there exists a compact subset $Q$ of $G$ and $\epsilon > 0$ such that, whenever a unitary representation $\rho$ of $G$ has a $(Q, \epsilon)$-invariant vector, then $\rho$, then $\rho$ has a non-zero invariant vector.

Definition 3.1 of Kazhdan's Property (T) is standard and mostly used in all kinds of context. For the purpose of this article, we use an equivalent definition of Kazhdan's Property (T) to finish the proof of main theorem.

**Definition 3.3.** Every unitary representation of $G$ that has an $(Q, \epsilon)$-invariant unit vector for any $\epsilon > 0$ and any compact subset $Q \subset G$, has no nonzero invariant vector.

Before we actually prove that Kazhdan's property (T) implies positiveness of the first eigenvalue of Laplacian, let us recall some results about Haar measure. More detailed material can be found in [7].

Let $G$ be a Lie group, and let $\mathfrak{g}$ be its Lie algebra. For $g \in G$, let $L_g : G \to G$ and $R_g : G \to G$ be the left and right translations $L_g(x) = gx$ and $R_g(x) = xg$. A smooth form $\omega$ on $G$ is *left invariant* if $L_g^*\omega = \omega$ for all $g \in G$, *right invariant* if $R_g^*\omega = \omega$ for all $g \in G$. Following theorem is proved in [7].

**Theorem 3.4.** *If $G$ is a Lie group of dimension $m$, then $G$ admits a nowhere-vanishing left-invariant smooth $m$ form $\omega$. Then $G$ can be oriented so that $\omega$ is positive, and $\omega$ defines a nonzero Borel measure $d\nu_G$ on $G$ that is left invariant in the sense that $d\nu_G(L(g)E) = d\nu_G(E)$ for all $g \in G$ and every Borel set $E$ in $G$.*

A nonzero Borel measure on $G$ invariant under left translation is called a *left Haar measure* on $G$ and its existence is guaranteed by Theorem 3.4.

It is known that a group $G$ is said to be unimodular if and only if left Haar measure is also right invariant. According to Corollary 8.31 in [7], semisimple Lie groups are always unimodular. Throughout this article, the Lie group $G$ is semisimple and then the left Haar measure $d\nu_G$ on $G$ is also right invariant.

**Theorem 3.5.** *Suppose that $G$ satisfies property (T) and $K$ is a maximal compact subgroup of $G$ which consists of elements fixed by Cartan involution, $\Gamma$ is a torsion-free, discrete subgroup of $G$ such that $\Gamma \setminus G/K$ is a noncompact locally symmetric manifold of infinite volume. Then the first eigenvalue $\lambda_0(\Gamma \setminus G/K)$ is bounded away from 0.*

Proof of the theorem depends on a few lemmas.

**Lemma 3.6.** *Suppose $G$ has Kazhdan's Property (T) and $K$ is a maximal compact subgroup of $G$, $\Gamma$ is a torsion free, discrete subgroup of $G$ such that $\Gamma \setminus G/K$ is noncompact locally symmetric manifold of infinite volume. Then the unitary representation $\rho : G \to \mathcal{U}(L^2(\Gamma \setminus G))$, defined as*

$$\rho(g)f(\bar{x}) := f(\bar{x}g)$$

*for all $g \in G$, has no invariant vectors.*

*Proof.* Assume the contrary: $\rho$ has an invariant vector. Let $f$ be the invariant vector in $L^2(\Gamma \setminus G)$ under representation $\rho : G \to \mathcal{U}(L^2(\Gamma \setminus G))$, i.e.

$$\rho(g)f = f, \text{ for all } g \in G$$

or

$$f(\bar{x}g) = f(\bar{x}), \text{ for all } g \in G$$

This implies that $f$ is a constant function on $\Gamma \setminus G$. Then $f \notin L^2$ due to the fact that $\Gamma \setminus G$ has infinite volume, contradicting to $f \in L^2(\Gamma \setminus G)$ $\square$

Throughout this article, $\bar{x}$ will always refer to the orbit of $\Gamma$ acting on $G$.

**Lemma 3.7.** *Suppose $u(\bar{x}) \in L^2(\Gamma \setminus G)$ and $\bar{u}(x)$ is the lift of $u(\bar{x})$ that is defined on $G$:*

$$\bar{u}(x) := u(\bar{x}).$$

*And $D$ is a fundamental region in $G$ with respect to $\Gamma$. Then for all $g \in G$, the function $\bar{u}(xg)$ is also $L^2$-integrable on $D$, i.e. $\bar{u}(xg) \in L^2(D)$.*

*Proof.* By definition of $\bar{u}(x)$, we have that

$$\bar{u}(xg) = u(\overline{xg}).$$

Since $\Gamma \cdot (xg) = (\Gamma x) \cdot g$, or equivalently, $\overline{xg} = \bar{x}g$. So

$$\bar{u}(xg) = u(\overline{xg}) = u(\bar{x}g).$$

Notice that $u(\bar{x}g)$ is in $L^2(\Gamma \setminus G)$ since it is the unitary representation of $G$ on $L^2(\Gamma \setminus G)$, and that $u(\bar{x}g) \in L^2(\Gamma \setminus G)$ is equivalent to $\bar{u}(xg) \in L^2(D)$ and $|d\bar{u}| \in L^2(D)$. We have completed the proof of lemma. □

**Lemma 3.8.** *Suppose both $h(\bar{x})$ and $dh(\bar{x})$ are $L^2$-integrable over $\Gamma \setminus G$, and $g$ is arbitrary element of $G$, then there exists a constant $C(g)$ that depends only on $g$, such that*

$$\int_{\Gamma \setminus G} |h(\bar{x}g) - h(\bar{x})|^2 d\nu_{\Gamma \setminus G} \leq C(g) ||dh||^2_{L^2(\Gamma \setminus G)} \tag{1}$$

*Proof.* Denote $u(x)$ as the lift of $h(\bar{x})$ from $\Gamma \setminus G$ to $G$, i.e,

$$u(x) := h(\bar{x}), \tag{2}$$

where $x \in G$ and $\bar{x} = \Gamma x$ that is the orbit of $x$ under action of $\Gamma$.

Let $D \subset G$ be a fundamental region of $G$ with respect to $\Gamma$. Since $h(\bar{x})$, $dh(\bar{x}) \in L^2(\Gamma \setminus G)$, we have $u(x)$, $du(x)$ are also $L^2$-integrable over any connected fundamental region $D \subset G$. Further more, we have following equality:

$$||dh||_{L^2(\Gamma \setminus G)} = ||du||_{L^2(D)}, \tag{3}$$

and

$$\int_{\Gamma \setminus G} |h(\bar{x}g) - h(\bar{x})|^2 d\nu_{\Gamma \setminus G} = \int_D |u(xg) - u(x)|^2 d\nu_G. \tag{4}$$

Let $g \in G$ taken arbitrarily, join $g$ and $e$ by a length-minimizing geodesic $g(t)$ of $G$ such that $g(1) = g$, and $g(0) = e$. Then $g(t)$ has constant speed c,

$$|\frac{d}{dt} g(t)| = \sqrt{\langle \frac{d}{dt} g(t), \frac{d}{dt} g(t) \rangle} = c. \tag{5}$$

Since the Riemannian metric $\langle \cdot, \cdot \rangle$ is left-invariant under left translation $L_x$ defined by any element $x \in G$, we have that

$$\langle \frac{d}{dt} xg(t), \frac{d}{dt} xg(t) \rangle = \langle \frac{d}{dt} g(t), \frac{d}{dt} g(t) \rangle = c^2. \tag{6}$$

By taking derivative of $u(xg(t))$ with respect to $t$ and chain rule, the following inequality is obtained.

$$|\frac{d}{dt} u(xg(t))| \leq |du(xg(t))| |\frac{d}{dt} xg(t)|. \tag{7}$$

Due to $|\frac{d}{dt}xg(t)| = c$, we have

$$|\frac{d}{dt}u(xg(t))| \leq c|du(xg(t))|. \tag{8}$$

On the other hand, we have the following estimate:

$$\begin{aligned}
|u(xg) - u(x)|^2 &= |\int_0^1 \frac{d}{dt}u(xg(t))dt|^2 \\
&\leq (\int_0^1 |\frac{d}{dt}u(xg(t))|dt)^2 \\
&\leq \int_0^1 |\frac{d}{dt}u(xg(t))|^2 dt \\
&\leq \int_0^1 c^2|du(xg(t))|^2 dt.
\end{aligned} \tag{9}$$

Integrating $|u(xg) - u(x)|^2$ over fundamental domain $D$ with respect to Haar measure $d\nu_G$ of $G$, we have

$$\begin{aligned}
\int_D |u(xg) - u(x)|^2 d\nu_G &= \int_G |u(xg) - u(x)|^2 \chi_D(x) d\nu_G \\
&\leq \int_G (\int_0^1 |du(xg(t))|^2 |\frac{d}{dt}g(t)|^2 dt) \chi_D(x) d\nu_G \\
&= \int_G \int_0^1 |du(xg(t))|^2 |\frac{d}{dt}g(t)|^2 \chi_D(x) dt d\nu_G \\
&= \int_0^1 \int_G |du(xg(t))|^2 |\frac{d}{dt}(g(t))|^2 \chi_D(x) d\nu_G dt \\
&= \int_0^1 |\frac{d}{dt}g(t)|^2 (\int_G |du(xg(t))|^2 \chi_D(x) d\nu_G) dt
\end{aligned} \tag{10}$$

Since $d\nu_G$ is left and right invariant Haar measure on $G$, the inside integral above has the following property,

$$\begin{aligned}
\int_G |du(xg(t))|^2 \chi_D(x) d\nu_G &= \int_G |du(x)|^2 \chi_D(xg^{-1}(t)) d\nu_G \\
&= \int_G |du(x)|^2 \chi_{Dg^{-1}(t)}(x) d\nu_G \\
&= \int_{Dg^{-1}(t)} |du(x)|^2 d\nu_G \\
&= ||du||^2_{L^2(Dg^{-1}(t))} \\
&= ||du||^2_{L^2(D)}.
\end{aligned} \tag{11}$$

Combining with (3.10), we have that

$$\int_D |u(xg) - u(x)|^2 d\nu_G \leq (\int_0^1 |\frac{d}{dt}g(t)|^2 dt) ||du||^2_{L^2(D)}, \tag{12}$$

or equivalently

$$\int_{\Gamma\backslash G} |h(\bar{x}g) - h(\bar{x})|^2 d\nu_{\Gamma\backslash G} \leq (\int_0^1 |\frac{d}{dt}g(t)|^2 dt) ||dh||^2_{L^2(\Gamma\backslash G)}. \tag{13}$$

Since $g(t)$ is geodesic, energy of the curve $g(t)$ and length $l(g(t))$ of $g(t)$ have equality relation as follows:

$$\int_0^1 |\frac{d}{dt}g(t)|^2 dt = \frac{1}{2}l^2(g(t)). \tag{14}$$

And $l(g(t))$ only depends on $g \in G$, let $C(g) = \frac{1}{2}l^2(g(t))$, we have proved that

$$\int_{\Gamma\backslash G} |h(\bar{x}g) - h(\bar{x})|^2 d\nu_{\Gamma\backslash G} \leq C(g)||dh||^2_{L^2(\Gamma\backslash G)}. \tag{15}$$

□

Now we prove Theorem 3.5:

*Proof.* We need to prove that $\lambda_0(\Gamma \setminus G/K)$ is bounded away from 0, or equivalently there exists constant $c > 0$, such that

$$\lambda_0 = \inf_{f\in C_c^\infty}\{\frac{\int_{\Gamma\backslash G/K}|df|^2 dvol}{\int_{\Gamma\backslash G/K}|f|^2 dvol}\} \geq c. \tag{16}$$

Normalising $L^2$-norm of $f$ such that $||f||_{L^2} = 1$, (3.16) is equivalent to

$$\lambda_0 = \inf_{f\in C_c^\infty}\{\int_{\Gamma\backslash G/K}|df|^2 dvol\} \geq c. \tag{17}$$

We prove this by assuming the contrary: $\lambda_0(\Gamma \setminus G/K) = 0$, that is to say for any positive number $c > 0$, there exists $f \in L^2(\Gamma \setminus G/K)$, such that

$$\int_{\Gamma\backslash G/K}|df|^2 dvol < c. \tag{18}$$

Now choose arbitrary compact subset $Q \subset G$, and arbitrary positive number $\epsilon > 0$. Notice that from the proof of Lemma 3.8, $l^2(g(t))$ is nothing but the square of arc length of the geodesic connecting $g$ and $e$. Since arc length $l(g(t))$ only depends on element $g$ continuously, $C(g)$ also depends on $g$ continuously. So ranging over the compact set $Q$, there exists a maximum value of $C(g)$ which can be defined as

$$C_M = \max_{g\in Q}\{C(g)\}. \tag{19}$$

And then for all $g \in Q$, result of Lemma 3.8 can be improved to

$$\int_{\Gamma\backslash G} |h(\bar{x}g) - h(\bar{x})|^2 d\nu_{\Gamma\backslash G} \leq C_M||dh||^2_{L^2(\Gamma\backslash G)} \tag{20}$$

By the assumption that $\lambda_0(\Gamma \setminus G/K) = 0$, there exists $f \in L^2(\Gamma \setminus G/K)$ that satisfies

$$\int_{\Gamma\backslash G/K}|df|^2 dvol < \frac{\epsilon}{C_M}, \tag{21}$$

where $C_M$ is the constant coming from (3.19) that only depends on $Q$.
Lift $f$ onto an $L^2$ function on $\Gamma \setminus G$, denoted as $\tilde{f}$, and it holds that

$$\int_{\Gamma\backslash G}|d\tilde{f}|^2 d\nu_{\Gamma\backslash G} < \frac{\epsilon}{C_M}. \tag{22}$$

Applying Lemma 3.8, we obtain that

$$\int_{\Gamma\backslash G} |\tilde{f}(xg) - \tilde{f}(x)|^2 d\nu_{\Gamma\backslash G} \leq C_M \int_{\Gamma\backslash G} |d\tilde{f}|^2 d\nu_{\Gamma\backslash G} < \epsilon, \tag{23}$$

or

$$||\rho(g)\tilde{f} - \tilde{f}||_{L^2} < \epsilon, \tag{24}$$

for all $g \in Q$.

So $\tilde{f}$ is a $(Q, \epsilon)$-invariant vector of the unitary representation

$$\rho : G \to \mathcal{U}(L^2(\Gamma \setminus G)).$$

Since the pair $(Q, \epsilon)$ is chosen arbitrarily, we conclude that if $\lambda_0(\Gamma \setminus G/K) = 0$, then $\rho$ is a unitary representation that has $(Q, \epsilon)$-invariant vectors for all compact set $Q \subset G$ and all $\epsilon > 0$. But the Kazhdan's property (T) of $G$ implies that this unitary representation $\rho$ has an invariant vector, contradicting to Lemma 3.6. □

## 4 Geometric superrigidity and Matsushima Formula

In this section we recall some results proved by N. Mok, Y.T. Siu and S.K. Yeung [16] and P. Pansu [15]. It is proved in [15] and [16] that there exists on a irreducible symmetric space that is neither real nor complex a 4-tensor $Q$ satisfying certain conditions. This 4-tensor is called a tensor of curvature type according to [16]. Bochner type formula for $Q$ is obtained in [16] via integration by parts. P. Pansu [15] generalizes the arguments to obtain the Matsushima formula for Hilbert space valued differential forms on $\Gamma \setminus G/K$.

According to [14] and [16], let $M$ be a Riemannian manifold with metric $(g_{ij})$. We say that a real 4-tensor $Q$ on $M$ is a tensor of curvature type if

$$\begin{aligned} Q(X,Y,Z,W) &= -Q(Y,X,Z,W) = Q(Z,W,X,Y) \\ Q(X,Y,Z,W) &+ Q(Y,Z,X,W) + Q(Z,X,Y,W) = 0 \end{aligned}$$

The actions of $Q$ on 1-forms and 2-tensors are defined as

$$Q(\alpha) = g_{km}Q^{ijkl}\alpha^m$$

and

$$\mathring{Q}\tau = g_{km}g_{ln}Q^{ikjl}\tau^{mn}$$

where $Q^{ijkl} = g^{ip}g^{jq}g^{kr}g^{ls}Q_{pqrs}$, $(g^{ij})$ is the inverse matrix of $(g_{ij})$.
The following proposition is stated in [15] and [16], and its proof can be found in [16].

**Proposition 4.1.** *Let $X$ be a irreducible symmetric space that is neither real nor complex hyperbolic. There exists on $X$ a tensor of curvature type $Q$ satisfying conditions:*
*1) $Q$ is parallel.*
*2) $\langle Q, R\rangle = 0$, where $\langle\cdot,\cdot\rangle$ means the pointwise inner product.*
*3) The quadratic form $\langle \mathring{Q}\tau, \tau\rangle$ is positive definite on traceless symmetric 2-tensors.*
*4) The inner product $\langle Q, T\rangle$ si nonpositive for any tensor $T$ of curvature type with nonpositive Riemannian sectional curvature in the case of rank $\geq 2$ and with nonpositive complexified sectional curvature in the rank 1 case.*

Following proposition can be found in [14]

**Proposition 4.2.** *Let $M$ be a compact Riemannian manifold, $R$ be the curvature tensor, $E$ be a vector fibre on $M$ with a metric and an orthogonal connection $D$. Let $\alpha$ be a 1-form on $M$ valued in $E$ and $Q$ be a tensor of curvature type that is parallel on $M$. We have that (by integration by parts)*

$$\int_M \langle \mathring{Q} D\alpha, D\alpha\rangle = \frac{1}{2}\int_M (\langle Q, \alpha^* R^E\rangle + \langle Q(\alpha), R(\alpha)\rangle). \tag{25}$$

Since $\langle Q(\alpha), R(\alpha)\rangle$ is proportional to $\langle Q, R\rangle|\alpha|^2$ if $M$ is locally symmetric and locally irreducible, $\langle Q(\alpha), R(\alpha)\rangle = 0$ from condition (2) in proposition 4.1. Condition (4) implies that $\langle Q, \alpha^* R^E\rangle \leq 0$ while condition (3) implies that $\langle \mathring{Q} D\alpha, D\alpha\rangle \geq 0$. Thus we have that

$$\int_M \langle \mathring{Q} D\alpha, D\alpha\rangle = 0 \tag{26}$$

In [15], Matsushima Formula was generalized to Hilbert space valued differential forms. Let $\Gamma$ be a discrete group of isometries on $X$, where $X$ is a Riemannian manifold. And $\rho : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation on a Hilbert space. Then $\rho$ gives rise to a flat bundle over $\Gamma \setminus X$ if we consider diagonal action of $\Gamma$ on $\mathcal{H} \times X$ defined as

$$(\gamma, \gamma)(h, x) = (\rho(\gamma)h, \gamma x) \tag{27}$$

Then $(\mathcal{H} \times X)/\Gamma$ is a flat bundle over $\Gamma \setminus X$ and a $\Gamma \setminus X$ valued differential form is a $\Gamma$-invariant $\mathcal{H}$-valued differential form on $X$. This kind of forms a called forms twisted by unitary representation $\rho$. Furthermore, $|\alpha|^2$ is a function on $X$ that is invariant under $\Gamma$. we define the $L^2$ norm of $\alpha$ as

$$||\alpha||^2_{L^2} = \int_{\Gamma \setminus X} |\alpha|^2 \tag{28}$$

Since the integration by parts procedure to derive (4.2) can be justified by cut-off argument [4], we obtain same result for $\Gamma \setminus X$:

$$\int_{\Gamma \setminus X} \langle \mathring{Q} D\alpha, D\alpha\rangle = 0 \tag{29}$$

The following lemma is proved by P. Pansu in [15].

**Lemma 4.3.** *Let $X$ be an irreducible symmetric space, that is not real or complex, $\Gamma$ a discrete group of isometry of $X$, $\rho$ a unitary representation of $\Gamma$. Let $\alpha$ be a 1-form on $\Gamma \setminus X$ twisted by $\rho$, $D\alpha$ its covariant derivative, we suppose $\alpha$, $d\alpha$, $\delta\alpha$ are in $L^2(\Gamma \setminus X)$. So there is a constant $C$ independent of $\rho$ such that*

$$||\alpha||^2 + ||D\alpha||^2 \leq C(||\delta\alpha||^2 + ||d\alpha||^2) \tag{30}$$

*Proof.* Let $X$ be a symmetric space of rank >1. $Q$ is a curvature tensor field parallel on $X$, satisfying $< Q, R >= 0$. Quadratic form $< Q(\xi), R(\xi) >$ on cotangent space is $K-$invariant, so proportional to a metric. If $< Q, R >= 0$, it is identically 0, so $< Q(\xi), R(\xi) >= 0$ for all $\xi \in T * X$.
According to Proposition 4.1, we can choose a parallel tensor field of curvature type $Q$ such that $< Q, R >= 0$. There exist constants $C'$ and $C''$, such that for all bilinear form $T$ on $TX$,

$$< T, T > \leq C'(|A(T)|^2 + |trT|^2) + C''\langle \mathring{Q} T, T\rangle \tag{31}$$

or $A(T)(u,v) = T(u,v) - T(v,u)$.
Let $\alpha$ be a 1-form on $\Gamma \setminus X$ twisted by $\rho$, as $d\alpha = A(D\alpha)$ and $\delta = tr(D\alpha)$, there comes

$$|D\alpha|^2 \leq C'(|d\alpha|^2 + |\delta\alpha|^2) + C''\langle \mathring{Q} D\alpha, D\alpha\rangle \tag{32}$$

where $\alpha$ and $D\alpha$ are $L^2$, the Matsushima formula is written as

$$\int_{\Gamma\setminus X} \langle \mathring{Q} D\alpha, D\alpha\rangle = 0. \tag{33}$$

whence integrate (4.8), we have

$$||D\alpha||^2 \leq C'(||d\alpha||^2 + ||\delta\alpha||^2)$$

Since $\alpha$, $D\alpha \in L^2$, the Bochner formula gives:

$$||D\alpha||^2 - ||\delta\alpha||^2 - ||d\alpha||^2 = L||\alpha||^2$$

where $L$ is a constant $> 0$. Then there exists constant $C$ such that $||\alpha||^2_{L^{1,2}} := ||\alpha||^2 + ||D\alpha||^2 \leq C(||d\alpha||^2 + ||\delta\alpha||^2)$. i.e.:

$$\begin{aligned}
||\alpha||^2 + ||D\alpha||^2 &\leq C'(||d\alpha||^2 + ||\delta\alpha||^2) + ||\alpha||^2 && (34)\\
&= C'(||d\alpha||^2 + ||\delta\alpha||^2) + \frac{||D\alpha||^2 - ||\delta\alpha||^2 - ||d\alpha||^2}{L} && (35)\\
&= (C' - \frac{1}{L})(||d\alpha||^2 + ||\delta\alpha||^2) + \frac{||D\alpha||^2}{L} && (36)\\
&\leq (C' - \frac{1}{L})(||d\alpha||^2 + ||\delta\alpha||^2) + (\frac{1}{L})C'(||d\alpha||^2) + ||\delta\alpha||^2 && (37)
\end{aligned}$$

where $C = C' - \frac{1}{L} + \frac{C'}{L}$. □

In particular, we have the following corollary immediately.

**Corollary 4.4.** *Let $\alpha$ be an $L^2$ harmonic 1-form on $\Gamma \setminus G/K$, then $\alpha = 0$.*

**Proposition 4.5.** *The codifferential $\delta$ is a 1-1 map on the space of closed $L^2$ 1-forms*

*Proof.* If $\alpha$ is closed, then $d\alpha = 0$. Assume that $\delta\alpha = 0$, form inequality of Lemma 4.4, there must be that $\alpha = 0$ □

**Corollary 4.6.** *The first $L^2$-cohomology of $\Gamma \setminus X$ twisted by $\rho$ vanishes,*

$$L^2H^1(\Gamma \setminus X, \rho) = 0 \tag{38}$$

*Proof.* We need to show that for any $L^2$ 1-form $\alpha$ satisfying $d\alpha = 0$, there exists an $L^2$ 0-form $\beta$ such that $d\beta = \alpha$.
Consider the following $L^2$ chain complex

$$L^2\Omega^0 \to L^2\Omega^1 \to L^2\Omega^2 \to ... \tag{39}$$

where $L^2\Omega^p$ is the space of $L^2$ differential $p$-forms.
Take $\beta = (\delta^{-1})^*\alpha$, where $\delta$ is the codifferential of the $L^2$ chain complex in (3.5) and $(\delta^{-1})^*$ is the adjoint of $\delta^{-1}$. $\delta^{-1}$ is well defined since $\delta$ is 1-1 on the space of closed $L^2$ 1-forms. We will show that $\beta$ is the $L^2$ 0-form such that $d\alpha = \beta$.

There are two norms for $\alpha \in L^2\Omega^1$, one is $||\alpha||^2_{L^2}$ defined as above, the other is $||\alpha||^2_{L^{1,2}}$ defined to be $||\alpha||^2_{L^2} + ||D\alpha||^2_{L^2}$, suppose that $D\alpha$ is $L^2$. By the definition of adjoint operator, we have following

$$\langle \delta\beta', \beta\rangle = \langle \delta\beta', (\delta^{-1})^*\alpha\rangle = \langle \delta^{-1}\delta\beta', \alpha\rangle = \langle \beta', \alpha\rangle \tag{40}$$

On the other hand, $\langle \delta\beta', \beta\rangle = \langle \beta', d\beta\rangle$.
Thus, $d\beta = \alpha$, and $\beta \in L^2\Omega^0$, which implies $L^2H^1(\Gamma \setminus X, \rho) = 0$ □

# 5 Proof of main theorem

## 5.1 Parabolic Ends of manifolds

Parabolicity of manifolds and their ends are extensively studied function theory on manifolds. In this section, we summarise some general results on parabolicity of non-compact manifolds and their ends and prove that parabolicity implies finiteness of volume of end. More detailed materials on these topics can be found in [8] and [11]

**Definition 5.1.** A complete manifold is called parabolic if it does not admit a positive Green's function. otherwise, it is called nonparabolic.

**Definition 5.2.** An end $E$ is non-parabolic if and only if it admits a positive harmonic function $f$ with

$$f = 1 \text{ on } \partial E$$
$$\liminf_{y\to\infty} f(y) < 1$$

**Definition 5.3.** An end E with respect to a compact subset $K \subset M$ is an unbounded connected component of $M \setminus K$. The number of ends with respect to $K$, denoted as $N_K(M)$ is the number of unbounded connected components of $M \setminus K$.

Obviously if $K_1 \subset K_2$, then $N_{K_1}(M) \leq N_{K_2}(M)$. By taking exhaustion $\{K_i\}$ of $M$, if the sequence $N_{K_i}(M)$ is bounded, we say the number of ends of $M$ is

$$N(M) = \max_{i\to\infty} N_{K_i}(M) \tag{41}$$

Note that $N(M)$ is independent of exhaustion $\{K_i\}$. In addition, we denote the boundary of $E$ as

$$\partial E = \partial K_1 \cap \bar{E} \tag{42}$$

**Definition 5.4.** The first eigenvalues of manifold $M$ and its end $E$ are defined to be

$$\lambda_0(M) = \inf_{f\in C_c^\infty(M)} \{\frac{\int_M |\nabla f|^2 dvol}{\int_M |f|^2 dvol}\}$$

and

$$\lambda_0(E) = \inf_{f\in C_c^\infty(E)} \{\frac{\int_E |\nabla f|^2 dvol}{\int_E |f|^2 dvol}\}.$$

The following result is immediate from definition.

**Proposition 5.5.** *If $\lambda_0(M) > 0$, then for all ends of $M$, $\lambda_0(E) > 0$.*

*Proof.* Choose arbitrary $f \in C_c^\infty(E)$, extending $f$ to $M$ by letting , we have that

$$\frac{\int_M |\nabla f|^2 dvol}{\int_M |f|^2} dvol = \frac{\int_E |\nabla f|^2 dvol}{\int_E |f|^2 dvol} \geq \lambda_0(M)$$

. Since $f$ lies in $C_c(M)$ automatically, and $f$ is chosen arbitrarily, it is obvious that $\lambda_0(M)$ is another lower bound of

$$\{\frac{\int_E |\nabla f|^2 dvol}{\int_E |f|^2 dvol}\}$$

. So $\lambda_0(M) \leq \lambda_0(E)$. □

Consider the boundary value problem on an end of complete Riemannian manifold $M$. Let $E$ be an end of $M$, $B(R)$ be a geodesic ball with radius $R$ centering at some point of $M$. Denote $E(R) = B(R) \cap E$. Suppose $f_i$ is the solution to the boundary value problem

$$\Delta f_i = 0 \text{ on } E(R_i)$$

with

$$f_i = 1 \text{ on } \partial E$$

and

$$f_i = 0 \text{ on } E \cap \partial B(R_i).$$

The following lemmas are proved in [8].

**Lemma 5.6.** *(Lemma 17.1 of [8]) Let $\Omega \subset M$ be a connected open subset of a complete manifold $M$. Suppose $\{f_i\}$ is a sequence of positive harmonic functions defined on $\Omega$. If there exists a point $p \in \Omega$ such that the sequence $f_i(p)$ is bounded, then after passing through a subsequence $f_i$ converges uniformly on compact subsets of $\Omega$ to a harmonic function $f$.*

**Lemma 5.7.** *(Lemma 20.6 of [8]) An end $E$ with respect to the compact set $B_p(R_0)$ is non-parabolic if and only if the sequence of positive harmonic functions $\{f_i\}$, defined on $E_p(R_i) = B_p(R_i) \cap E$ for $R_0 < R_1 < ... <\to \infty$. satisfying*

$$f_i = 1 \text{ on } \partial E$$

*and*

$$f_i = 0 \text{ on } \partial B_p(R_i) \cap E$$

*converges uniformly on compact subsets of $E \cup \partial E$ to a non-constant function $f$.*

The following proposition is a special case of Lemma 20.10 of [8].

**Proposition 5.8.** *Let $E$ be a parabolic end of a complete Riemannian manifold. Suppose the spectrum $\lambda_0(E) > 0$. Then $E$ has finite volume.*

*Proof.* According to Lemma 17.1 and Lemma 20.6 of [8], the solution $f_i$ of above boundary value problem converges to a constant function provided $E$ is parabolic.
Fix $R_0$ large enough that $E(R_0) = B(R_0) \cap E \neq \emptyset$, $R > R_0$ let $\phi$ be a nonnegative cutoff function satisfying

$$\phi = 1 \text{ on } E(R) - E(R_0)$$

and

$$\phi = 0\text{on } \partial E$$

with

$$|\nabla\phi| \leq C_1.$$

Since $\lambda_(E) > 0$, we have

$$0 < \lambda_0(E) \leq \frac{\int_{E(R)} |\nabla(\phi f_i)|^2 dVol}{\int_{E(R)} |\phi f_i|^2 dVol},$$

or equivalently

$$\int_{E(R)} |\phi f_i|^2 dVol \leq \frac{1}{\lambda_0(E)} \int_{E(R)} |\nabla(\phi f_i)|^2 dVol.$$

Since

$$\begin{aligned}
&\int_{E(R)} |\nabla(\phi f_i)|^2 dVol \\
&= \int_{E(R)} |\nabla\phi|^2 f_i^2 dVol + 2\int_{E(R)} \phi f_i \langle \nabla\phi, \nabla f_i\rangle dVol + \int_{E(R)} \phi^2 |\nabla f_i| dVol \\
&= \int_{E(R)} |\nabla\phi|^2 f_i^2 dVol + \frac{1}{2}\int_{E(R)} \langle \nabla(\phi)^2, \nabla(f_i)^2\rangle dVol + \int_{E(R)} \phi^2 |\nabla f_i| dVol \\
&= \int_{E(R)} |\nabla\phi|^2 f_i^2 dVol,
\end{aligned}$$

we have that for any $i, j$, $0 < i < j$,

$$\int_{E(R_i)-E(R_0)} f_j^2 dVol \leq \frac{C_1}{\lambda_0(E)} \int_{E(R_0)} f_j^2 dVol.$$

$f_j$ converges to a constant, so letting $j \to \infty$, we have that

$$Vol(E(R_i)) - Vol(E(R_0)) \leq C_2 Vol(E(R_0)).$$

Letting $i \to \infty$, we conclude that $E$ has finite volume. □

**Lemma 5.9.** *$\Gamma \setminus G/K$ has at most one non-parabolic end. Furthermore, it has at most one end with infinite volume.*

*Proof.* For our purpose, let us assume that $\Gamma \setminus G/K$ has at least two non-parabolic ends, otherwise, there is nothing to prove. For the ball $B_p(R_0)$ centering at $p$ of $\Gamma \setminus G/K$, take $R_0$ sufficiently large so that $(\Gamma \setminus G/K) \setminus B_p(R_0)$ has at least two disjoint non-parabolic ends $E_1$ and $E_2$. For $R_i > R_0$, we can solve the following boundary value problem on $E_1$ to get $f_{R_i}$,

$$\Delta f_{R_i} = 0 \text{ on } B_p(R_i)$$

$$f_{R_i} = 1 \text{ on } E_1 \cap \partial B_p(R_i)$$

$$f_{R_i} = 0 \text{ on } \partial B_p(R_i) \setminus E_1$$

Since $E_1, E_2$ are non-parabolic, there is a subsequence of $\{f_i\}$ converging to a harmonic function $f$ defined on $\Gamma \setminus G/K$ such that

$$\sup_{\Gamma\setminus G/K} f = \sup_{E_1} f = 1 \tag{43}$$

$$\inf_{\Gamma\setminus G/K} f = \inf_{E_i} f = 0, \quad i \neq 1. \tag{44}$$

According to arguments of Li-Tam in [9] and [10], let $B_p(R_0)$ be a ball of $M = \Gamma \setminus G/K$, the function $f$ constructed above is harmonic on $M - B_p(R_0)$. Lemma 1.4 of [10] gives a sequence of harmonic functions $\{f_i\}$ on $B_p(R_i) - B_p(R_0)$ with $f_i = f$ on $\partial B_p(R_0)$ and $f_i = 0$ on $\partial B_p(R_i)$ such that $f = \lim f_i$. Then

$$\int_{B_p(R_i)-B_p(R_0)} |\nabla f_i|^2 dVol + \int_{B_p(R_i)-B_p(R_0)} f_i \Delta f_i dVol \tag{45}$$

$$= \int_{\partial(B_p(R_i)-B_p(R_0))} f_i \frac{\partial f_i}{\partial r} dS \tag{46}$$

$$= \int_{\partial B_p(R_0)} f_i \frac{\partial f_i}{\partial r} dS \tag{47}$$

For $i < j$,

$$\int_{B_p(R_i)-B_p(R_0)} |\nabla f_j|^2 dVol \leq \int_{B_p(R_j)-B_p(R_0)} |\nabla f_j|^2 dVol \tag{48}$$

$$= \int_{B_p(R_0)} f_j \frac{\partial f_j}{\partial r} dS \tag{49}$$

$\int_{B(R_0)} f_j \frac{\partial f_j}{\partial r} dS$ is uniformly bounded according to [8]. Consider the function $|\nabla \log f_i|^2$, $|\nabla \log f_i|^2$ is uniformly bounded on $\partial B(R_0)$.

Let $x_0 \in \partial B(R_0)$ be the maximum point of $|\nabla \log f_i|^2$, and denote $h = \log f_i$, then the strong maximum principle yields

$$\frac{\partial |\nabla h|^2}{\partial r}(x_0) < 0 \tag{50}$$

Choose orthonormal frame $\{e_1, ..., e_n\}$ at $x_0$, such that $e_1 = \frac{\nabla h}{|\nabla h|} = -\frac{\partial}{\partial r}$, which is the outward normal vector at $\partial B(R_0)$.

Since $|\nabla h| e_1 = \nabla h$, we have

$$\langle |\nabla h| e_1, e_1 \rangle = \langle \nabla h, e_1 \rangle = \frac{\partial h}{\partial x_1} = |\nabla h| \tag{51}$$

Then

$$\begin{aligned}
\frac{\partial}{\partial r} |\nabla h|^2 &= \frac{\partial}{\partial r} (\frac{\partial h}{\partial x_1})^2 \\
&= -\frac{\partial}{\partial r} [-(\frac{\partial h}{\partial x_1})^2] \\
&= \langle -e_1, \nabla (\frac{\partial h}{\partial x_1})^2 \rangle \\
&= \langle -e_1, \frac{\partial}{\partial x_1} (\frac{\partial h}{\partial x_1})^2, ..., \frac{\partial}{\partial x_n} (\frac{\partial h}{\partial x_n})^2 \rangle \\
&= -\frac{\partial}{\partial x_1} (\frac{\partial h}{\partial x_1})^2 \\
&= -2 \frac{\partial h}{\partial x_1} \frac{\partial^2 h}{\partial x_1^2} \\
&= -2 |\nabla h| \frac{\partial^2 h}{\partial x_1^2}
\end{aligned}$$

From the inequality (5.10), we have that

$$-2|\nabla h|\frac{\partial^2 h}{\partial x_1^2} < 0 \tag{52}$$

Since the Laplacian of $h$ is given by

$$\Delta h = -|\nabla h|^2 = \frac{\partial^2}{\partial x_1^2} + H\frac{\partial h}{\partial x_1} \tag{53}$$

where $H$ is the mean curvature of $\partial B(R_0)$ with respect to $e_1$. Now we obtained

$$|\nabla h|^3 + H|\nabla h|^2 < 0 \tag{54}$$

Thus the minimum value of $H$ must be negative. Denote it as $H_0 = \min_{\partial E} H < 0$, and we obtained the inequality that

$$|\nabla h| \leq -H \leq -H_0 \tag{55}$$

and

$$|\nabla h|^2 \leq H_0^2 \tag{56}$$

We can smooth out $\partial B(R_0)$ so that $|H_0| < \infty$, and this proves that $|\nabla \log f_i|$ is uniformly bounded on $\partial B(R_0)$. Since $f_i = 1$ on $\partial B(R_0)$, we can conclude that $|\frac{\partial f_i}{\partial r}|$ is uniformly bounded on $\partial B(R_0)$ by chain rule:

$$|\nabla \log f_i| = \frac{|\nabla f_i|}{|f_i|} = |\nabla f_i| = |\frac{\partial f_i}{\partial r}| \tag{57}$$

Let $j \to \infty$ and $i \to \infty$, we obtain that

$$\int_{M-B_p(R_0)} |\nabla f|^2 dVol \leq \int_{\partial B_p(R_0)} f\frac{\partial f}{\partial r} dS < \infty \tag{58}$$

Thus $f$ is of finite Dirichlet energy and for each non-parabolic end, we can get linearly independent harmonic functions on it. These linearly independent harmonic functions including constant functions form a linear space whose dimension is bounded below by the number of non-parabolic ends. Furthermore, according to [11], the exterior differential of a harmonic function with finite Dirichlet integral is an $L^2$ harmonic 1-form. We have

$$\dim \mathcal{H}^1_{L^2}(\Gamma \setminus G/K) + 1 \geq \text{number of non-parabolic ends.}$$

Since $\lambda_0(\Gamma \setminus G/K) > 0$, we have

$$\dim \mathcal{H}^1_{L^2}(\Gamma \setminus G/K) + 1 \geq \text{number of ends with infinite volumes.}$$

It is proved that the space of $L^2$-harmonic 1-form of $\Gamma \setminus G/K = 0$, thus $\dim \mathcal{H}^1_{L^2}(\Gamma \setminus G/K) = 0$, we conclude that $\Gamma \setminus G/K$ has at most one end of infinite volume. $\square$

### 5.2 Geometrical Finiteness

To proof Theorem 1.3, one needs some results of geometrical finiteness with negative curvature. We recall B. Bowditch's classification on geometrical finiteness of variable negative curvature. For more details on the work of B. Bowditch, please refer [2].

A *Hadamard manifold* is a complete, simply connected Riemannian manifold of nonpositive curvature. By a pinched *Hadamard manifold*, we shall mean a Hadamard manifold of pinched negative curvature. Let $\Gamma$ be a discrete group of isometries of a pinched Hadamard manifold, $X$. To the orbifold $\Gamma \setminus X$, we adjoin the quotient $\Gamma \setminus \Omega$, where $\Omega$ is the domain of discontinuity of the ideal sphere at infinity. Then $M_C(\Gamma) = \Gamma \setminus (X \cup \Omega)$ is an orbifold with boundary. The main result of [2] is summarized as follows.

**Theorem 5.10.** *Let $X$ be a Hadamard manifold with pinched negative curvature, $\Gamma$ be a discrete subgroup of isometry of $X$. Then the following definitions of geometrical finiteness of $\Gamma$ are equivalent.*

*F1. $M_C(\Gamma)$ has finitely many ends, each a parabolic end.*

*F2. The limit set $\Lambda$ consists entirely of conical limit points and bounded parabolic fixed points.*

*F4. For some positive number $\epsilon \in (0, \epsilon(n,k))$, core$(M)\cap$thick$_\epsilon(M)$ is compact, where $\epsilon(n,k)$ is the Margulis constant, core$(M)$ is the convex core of $M$.*

*F5. There is a bound on the orders of every finite subgroup of $\Gamma$, and for some $\eta > 0$, $N_\eta$core$(M)$ has finite volume, where $N_\eta$core$(M)$ is the $\eta$-neighbourhood.*

We now prove Theorem 1.3:

*Proof.* Let $A$ be the compact component of $\Gamma \setminus \Omega$, $e(A)$ be the end corresponding to $A$ and $E(A)$ be a neighborhood of $e(A)$. By Theorem 1.1, we have that $\Gamma \setminus H^n_{\mathbb{K}} - E(A)$ is of finite volume. Denote $M = \Gamma \setminus H^n_{\mathbb{K}} - E(A)$. Let $X$ be the Riemannian universal covering of $M$ such that $M = \Gamma' \setminus X$ where $\Gamma' \cong \pi_1 M$. Since $M$ has finite volume, $\Gamma'$ is "F5", so it is "F1" which completes the proof. $\square$

## References


[1] B. Bekka, P. de la Harpe and A. Valette, Kazhdan's Property (T), Cambridge University Press, 2008.

[2] B. H. Bowditch, Geometrical finiteness with variable negative curvature. *Duke Math. J.* **77**(1995), no.1, 229-274.

[3] H. Cao, Y. Shen and S. Zhu, The structure of stable minimal hypersurfaces in $\mathbb{R}^{n+1}$, *Math. Res. Let.* **4** (1997), p. 637-644.

[4] G. Carron, E. Pedon, On the differential form spectrum of hyperbolic manifolds Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 3 (4) (2004), pp. 705?747.

[5] K. Corlette, Hausdorff dimensions of limit sets I, *Invent. Math.* **102** (1990), 521-542.

[6] K. Corlette and A. Iozzi, Limit sets of discrete groups of isometries of exotic hyperbolic spaces, *Trans. Amer. Math. Soc.* **351** (1999), no. 4, p. 1507-1530.

[7] A. Knapp, Lie Groups Beyond an Introduction, Progress in Mathematics, Volume 140.

[8] P. Li, Geometric Analysis, Cambridge University Press, 2012.

[9] P. Li, L. Tam, Positive harmonic functions on complete manifolds with non-negative curvature outside a compact set, *Ann. of Math.* (2) **125** (1987) 171-207.

[10] P. Li, L. Tam, Harmonic functions and the structure of complete manifolds, *J. Differential Geom.* **35** (1992) 359-383.

[11] P. Li and J. Wang, Complete manifolds with positive spectrum, *J. Differential Geom.* **58** (2001), no. 3, 501-534.

[12] T. Napier and M. Ramachandran, Structure theorems for complete Kahler manifolds and applications to Lefschetz type theorems, *Geom. Funct. Anal.* **5** (1995), 809-851.

[13] T. Napier and M. Ramachandran, The $L^2\partial$-method, weak Lefschetz theorems, and the topology of Kahler manifolds, *J. Amer. Math. Soc.*, **11**, 2 (1998), 375-396.

[14] P. Pansu, Sous-groupes discrets des groupes de Lie:rigidit, arithmticit, *Sminaire N. Bourbaki*, 1993-1994, exp. No 778,p 69-105.

[15] P. Pansu, Formules de Matrsushima, de Garland et proprit (T) pour des groupes agissant sur des espaces symtriques, *Bull. Soc. math. France*, *126*, (1998), p. 107-139.

[16] N. Mok, Y. Siu, S. Yeung, Geometric superrigidity, *Invent. Math.*, **113**, (1993), p. 57-83.

[17] D. Vogan, Cohomology and group representations, Proceedings of Symposia in Pure Mathematics **61**. American Mathematical Society, Providence, RI (1997).