

Conics and Quadratic Forms in Kepler and Newton

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The Keplerian Origin

The story of quadratic forms in physics begins not with pure algebra, but with a cosmological quest. Between 1609 and 1619, Johannes Kepler announced three laws of planetary motion, the first being:

Planets move in ellipses, with the Sun at one focus.

In Cartesian coordinates, it takes the form of a positive-definite quadratic equation:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{or} \quad Ax^2 + Bxy + Cy^2 = 1, \quad \text{with } B^2 - 4AC < 0$$

Kepler did not possess this algebraic language (he worked geometrically), but his ellipses are the first great empirical instantiation of what we now call a quadratic form.

Newton's *Principia*

In 1687, Isaac Newton provided the *why*. In Book I of the *Philosophiæ Naturalis Principia Mathematica*, Proposition 11, he proved:

Under an inverse-square central force, a body traces a conic section (ellipse, parabola, or hyperbola).

The orbit satisfies a second-order differential equation whose solutions are precisely zero-sets of quadratic forms. Writing the orbit in polar coordinates $r(\theta)$ leads to the conic equation:

$$r(\theta) = \frac{l}{1 + e \cos \theta}$$

In Cartesian coordinates, clearing denominators yields the general quadratic form:

$$\alpha x^2 + \beta xy + \gamma y^2 + \delta x + \epsilon y + \zeta = 0$$

The discriminant $\beta^2 - 4\alpha\gamma$ determines the type:

- ▶ Ellipse (negative): bound orbits (planets, moons)
- ▶ Parabola (zero): escape velocity threshold
- ▶ Hyperbola (positive): unbound orbits (some comets)

Newton's genius was to show that a single force law produces all three algebraic families. Thus, quadratic forms became the algebraic signature of gravitational motion.

Kepler's laws as descriptive principles

1. Ellipse law: orbits are ellipse with the sun at one focus.
2. Equal area law: a line from planet to sun sweeps equal area in equal time (conservation of angular momentum).
3. The square of a planet's orbital period is proportional to the cube of the length of the semi-major axis of its orbit, $T^2 \propto a^3$.

Newton's inverse square force

Newton showed that Kepler's equal area law implies a central force directed at the sun, and Kepler's ellipse law forces that central force to follow an inverse-square relation:

$$F(r) = \frac{GMm}{r^2}$$

How Newton's Law of Universal Gravitation explains Kepler's Laws

Surprisingly, the first of Kepler's laws is the most difficult to prove beginning with Newton's assumption of inverse-square gravitation. Newton himself did it with a geometrical argument, famously difficult to follow.

Find the following proof online:

Prove that the orbit is an ellipse if the gravitational force is inverse square.

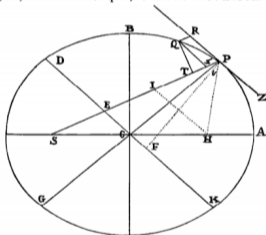
SECTION III

The motion of bodies in eccentric conic sections.

PROPOSITION XI. PROBLEM VI

If a body revolves in an ellipse; it is required to find the law of the centripetal force tending to the focus of the ellipse.

Let S be the focus of the ellipse. Draw SP cutting the diameter DK of the ellipse in E, and the ordinate Qv in x; and complete the parallelogram QxPR. It is evident that EP is equal to the greater semiaxis AC: for drawing HI from the other focus H of the ellipse parallel to EC, because CS, CH are equal, ES, EI will be also equal; so that EP is the half-sum of PS, PI,



that is (because of the parallels HI, PR, and the equal angles IPR, HPZ), of PS, PH, which taken together are equal to the whole axis 2AC. Draw QT perpendicular to SP, and putting L for the principal latus rectum of the ellipse (or for $\frac{2BC^2}{AC}$), we shall have

$$L \cdot QR : L \cdot Pv = QR : Pv = PE : PC = AC : PC,$$

$$\text{also, } L \cdot Pv : Gv \cdot Pv = L : Gv, \text{ and, } Gv \cdot Pv : Qv^2 = PC^2 : CD^2.$$

By Cor. II, Lem. VII, when the points P and Q coincide, $Qv^2 = Qx^2$, and

