Differential Geometry of Curves and Surfaces

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Curves

Definition

We say a vector function $\mathbf{f} : (a, b) \to \mathbb{R}^3$ is C^k if $\mathbf{f}$ and its first k derivatives, $\mathbf{f}'$, $\mathbf{f}''$, ... $\mathbf{f}^k$, exists and are all continuous. A *parametrized curve* is a C^3 map $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ for some interval $I = (a, b)$ or $[a, b]$. We say $\boldsymbol{\alpha}$ is regular if $\boldsymbol{\alpha}'(t) \neq 0$ for all $t \in I$.

- The velocity of the curve (imagine some particle moving along $\boldsymbol{\alpha}$) is

$$\alpha'(t) = \frac{d\boldsymbol{\alpha}}{dt} = \lim_{h \to 0} \frac{\boldsymbol{\alpha}(t+h) - \boldsymbol{\alpha}(t)}{h}$$

- The velocity vector $\boldsymbol{\alpha}'(t)$ is tangent to the curve at $\boldsymbol{\alpha}(t)$ and its length $\|\boldsymbol{\alpha}'(t)\|$ is the speed of the curve.

The distance a particle travels is the integral of its speed,

Proposition

Let $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^3$ *be a piecewise* C^1 *parametrized curve. Then*

$$\mathit{length}(\boldsymbol{\alpha}) = \int_a^b \|\boldsymbol{\alpha}'(t)\| \, dt.$$

Define $s(t)$ to be the arclength of the curve $\boldsymbol{\alpha}$ on the interval $[a, t]$. If $\boldsymbol{\alpha}'(t) = 1$ for all $t \in [a, b]$, then $s(t) = t - a$. We say the curve $\boldsymbol{\alpha}$ is *parametrized by arclength* if $s(t) = t$ for all t.

Existence of arclength parametriziation

Suppose $\boldsymbol{\alpha}$ is a regular curve, i.e., $\|\boldsymbol{\alpha}'(t)\| > 0$ for all t, then the arclength function $s(t) = \int_a^t \|\boldsymbol{\alpha}'(u)du\|$ is an increasing function (since $s'(t) = \|\boldsymbol{\alpha}'(t)\| > 0$), and therefore has a differentiable inverse function $t = t(s)$. Then the parametrization

$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(t(s))$$

is parametrization by arclength. (Ex. verify)

Example

Consider the helix $\boldsymbol{\alpha}(t) = (a\cos t, a\sin t, bt)$. Calculate $\boldsymbol{\alpha}'(t)$, $\|\boldsymbol{\alpha}'(t)\|$, and reparametrize $\boldsymbol{\alpha}$ by arc-length.

$$\boldsymbol{\alpha}'(t) = (-a\sin t, a\cos t, b)$$

$$\|\boldsymbol{\alpha}'(t)\| = \sqrt{a^2+b^2}$$

The arclength formula

$$s(t) = \int_0^t \|\boldsymbol{\alpha}'(u)\|\, du = \int_0^t \sqrt{a^2+b^2} du = \sqrt{a^2+b^2}t$$

gives the inverse function

$$t(s) = \frac{1}{\sqrt{a^2+b^2}} s$$

Therefore, the reparametrization by s is

$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(t(s)) = (a\cos\frac{1}{\sqrt{a^2+b^2}}s, a\sin\frac{1}{\sqrt{a^2+b^2}}s, \frac{b}{\sqrt{a^2+b^2}}s).$$

Local Theory: Frenet Frame

The following result is from vector calculus.

Lemma

Suppose $\mathbf{f}, \mathbf{g} : (a, b) \to \mathbb{R}^3$ *are differentiable and satisfy* $\mathbf{f}(t) \cdot \mathbf{g}(t) = const$ *for all* t. *Then* $\mathbf{f}'(t) \cdot \mathbf{g}(t) = -\mathbf{f}(t) \cdot \mathbf{g}'(t)$. *In particular* $\|\mathbf{f}(t)\| = const$ *if and only if* $\mathbf{f}(t) \cdot \mathbf{f}'(t) = 0$ *for all* t.

Using this lemma repeatedly, we can construct the *Frenet frame* of suitable regular curves.

- We assume throughout that the curve $\boldsymbol{\alpha}$ is parametrized by arclength. Then $\boldsymbol{\alpha}'(s)$ is the *unit tangent vector* to the curve, which we denote by $\mathbf{T}(s)$.
- Since $\mathbf{T}(s)$ has constant length, $\mathbf{T}(s)$ will be orthogonal to $\mathbf{T}(s)$.
- Assuming $\mathbf{T}'(s) \neq \mathbf{0}$, define the *principal normal vector* $\mathbf{N}(s) = \frac{\mathbf{T}'(s)}{\|\mathbf{T}'(s)\|}$ and the *curvature* $\kappa(s) = \|\mathbf{T}'(s)\|$, and we have

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s).$$

- If $\kappa \neq 0$, define the *binormal vector* $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$.

Then $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ form a right-handed orthonormal basis for $\mathbb{R}^3$. $\mathbf{N}'(s)$ must be a linear combination of $\mathbf{T}(s), \mathbf{N}(s)$ and $\mathbf{B}(s)$. Suppose there exist $a(s)$, bs and $c(s)$ such that

$$\mathbf{N}'(s) = a(s)\mathbf{T}(s) + b(s)\mathbf{N}(s) + c(s)\mathbf{B}(s)$$

note that

$$\mathbf{N}'(s) \cdot \mathbf{N}(s) = b(s) \left\| \mathbf{N}(s) \right\|^2 = b(s) = 0.$$

$$\mathbf{N}'(s) \cdot \mathbf{T}(s) = a(s) = -\mathbf{T}'(s)\mathbf{N}(s) = -\kappa(s)\mathbf{N}(s) \cdot \mathbf{N}(s) = -\kappa(s)$$

The component coefficient $c(s)$ is called the *torsion* (denoted $\tau(s)$) of the curve and can be computed from

$$\tau(s) = \mathbf{N}'(s) \cdot \mathbf{B}(s),$$

and this gives the linear expansion of $\mathbf{N}'(s)$:

$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s).$$

Similarly, $\mathbf{B}'(s)$ must be a linear combination of $\mathbf{T}(s)$, $\mathbf{N}(s)$ and $\mathbf{B}(s)$, we can assume

$$\mathbf{B}'(s) = a(s)\mathbf{T}(s) + b(s)\mathbf{N}(s) + c(s)\mathbf{B}(s).$$

Apparently,

$$\mathbf{B}'(s) \cdot \mathbf{T}(s) = -\mathbf{T}'(s) \cdot \mathbf{B}(s) = \kappa(s)\mathbf{N}(s) \cdot \mathbf{B}(s) = 0 = a(s).$$

And

$$b(s) = \mathbf{B}'(s) \cdot \mathbf{N}(s) = -\mathbf{N}'(s) \cdot \mathbf{B}(s) = -\tau(s)$$

In the end, $c(s)$ can be computed from

$$\mathbf{B}'(s) \cdot \mathbf{B}(s) = c(s) = 0,$$

and we have

$$\mathbf{B}'(s) = -\tau(s)\mathbf{N}(s)$$

In summary, the Frenet formulas of a curve is given by

$$\begin{aligned}
\mathbf{T}'(s) &= \kappa(s)\mathbf{N}(s) \\
\mathbf{N}'(s) &= -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s) \\
\mathbf{B}'(s) &= -\tau(s)\mathbf{N}(s)
\end{aligned}$$

Surfaces

Parametrized surfaces

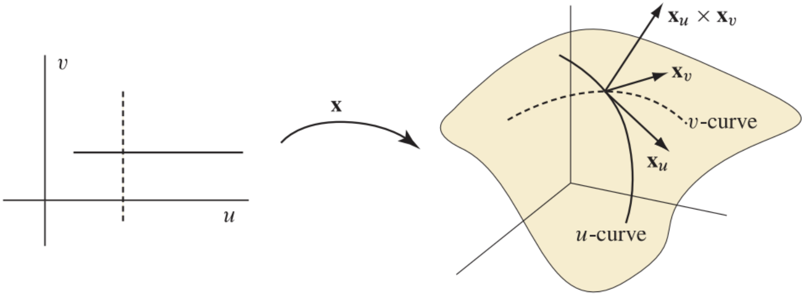
- Let U be an open set in $\mathbb{R}^2$. A function $\mathbf{f} : U \to \mathbb{R}^m$ is called C^1 if $\mathbf{f}$ and its partial derivatives $\frac{\partial \mathbf{f}}{\partial u}$ and $\frac{\partial \mathbf{f}}{\partial v}$ are all continuous.
- We will use (u, v) as coordinates in parameter space, and (x, y, z) as coordinates in $\mathbb{R}^3$.
- If $\mathbf{f}$ is C^2, then $\frac{\partial^2 \mathbf{f}}{\partial u \partial v} = \frac{\partial^2 \mathbf{f}}{\partial v \partial u}$.
- A *regular parametrization* of a subset $M \subset \mathbb{R}^3$ is a one-to-one function

$$\mathbf{x} : U \to M \subset \mathbb{R}^3, \quad \text{so that} \quad \mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$$

for some open set $U \subset \mathbb{R}^2$. ($\mathbf{x}_u$ and $\mathbf{x}_v$ are short for $\frac{\partial \mathbf{x}}{\partial u}$ and $\frac{\partial \mathbf{x}}{\partial v}$)

u-curve and v-curve

Consider the curves on M obtained by fixing $v = v_0$ and varying u, it is called a u-curve, and obtained by fixing $u = u_0$ and varying v, it is called a v-curve. At the point $p = \mathbf{x}(u_0, v_0)$, $\mathbf{x}_u(u_0, v_0)$ is tangent to the u-curve and $\mathbf{x}_v(u_0, v_0)$ is tangent to the v-curve.



The tangent plane of a surface is defined following that of general manifold, in parametrized surfaces, it can be defined concretely.

Definition

Let M be a regular parametrized surface, and let $p \in M$. Then choose a regular parametrization $\mathbf{x} : U \to M \subset \mathbb{R}^3$ with $p = \mathbf{x}(u_0, v_0)$. We define the tangent plane of M at p to be the subspace T_pM spanned by $\mathbf{x}_u$ and $\mathbf{x}_v$ evaluated at (u_0, v_0).

Example

- Graph of a function $f : U \to \mathbb{R}$, $z = f(x, y)$, is parametrized by $\mathbf{x}(u, v) = (u, v, f(u, v))$. Note that $\mathbf{x}_u \times \mathbf{x}_v = (-f_u, -f_v, 1) \neq \mathbf{0}$, so this is always a regular parametrization.
- The helicoid is the surface formed by drawing horizontal rays from the axis of the helix $\boldsymbol{\alpha}(t) = (\cos t, \sin t, bt)$ to points on the helix:

$$\mathbf{x}(u, v) = (u \cos v, u \sin v, bv), \quad u > 0, v \in \mathbb{R}.$$

Definition (First fundamental form)

In a parametrization of a surface $M \subset \mathbb{R}^3$, the first fundamental form, is defined as $I_p(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$, for $\mathbf{u}, \mathbf{v} \in T_pM$. Take $\{\mathbf{x}_u, \mathbf{x}_v\}$ as a natural basis, we also define

$$E = I_p(\mathbf{x}_u, \mathbf{x}_v) = \mathbf{x}_u \cdot \mathbf{x}_v \tag{1}$$

$$F = I_p(\mathbf{x}_u, \mathbf{x}_v) = \mathbf{x}_u \cdot \mathbf{x}_v = \mathbf{x}_v \cdot \mathbf{x}_u = I_p(\mathbf{v}_v, \mathbf{x}_u) \tag{2}$$

$$G = I_p(\mathbf{x}_v, \mathbf{x}_u) = \mathbf{x}_v \cdot \mathbf{x}_v \tag{3}$$

Often it is convenient to write it as entries of a symmetric matrix

$$I_p = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

Length of tangent vectors

Then given tangent vectors $\mathbf{u} = a\mathbf{x}_u + b\mathbf{x}_v$ and $\mathbf{v} = c\mathbf{x}_u + d\mathbf{x}_v \in T_pM$, we have

$$\mathbf{u} \cdot \mathbf{v} = I_p(\mathbf{u}, \mathbf{v}) = (a\mathbf{x}_u + b\mathbf{x}_v) \cdot (c\mathbf{x}_u + d\mathbf{x}_v) = E(ac) + F(ad + bc) + G(bd).$$

in particular, the length of $\mathbf{u}$, $\|\mathbf{u}\|^2 = I_p(\mathbf{u}, \mathbf{u}) = Ea^2 + 2Fab + Gb^2$.

I_p is an invariant

Let M and M' are surfaces. We say they are *locally isometric* if for each $p \in M$ there are a regular parametrization $\mathbf{x} : U \to M$ with $\mathbf{x}(u_0, v_0) = p$ and a regular parametrization $\mathbf{x}^* : U \to M^*$ (using the same domain U) with the property that $I_p = I_p^*$ whenever $p = \mathbf{x}(u, v)$ and $p^* = \mathbf{x}^*(u, v)$ for some $(u, v) \in U$.

I_p encodes surface area

The infinitesimal area of the parallelepiped spanned by $\mathbf{x}_u, \mathbf{x}_v$ is $\|\mathbf{x}_u \times \mathbf{x}_v\|$ (?), and the surface area over U can be computed by

$$\int_U \|\mathbf{x}_u \times \mathbf{x}_v\| \, dudv = \int_U \sqrt{EG - F^2} dudv.$$

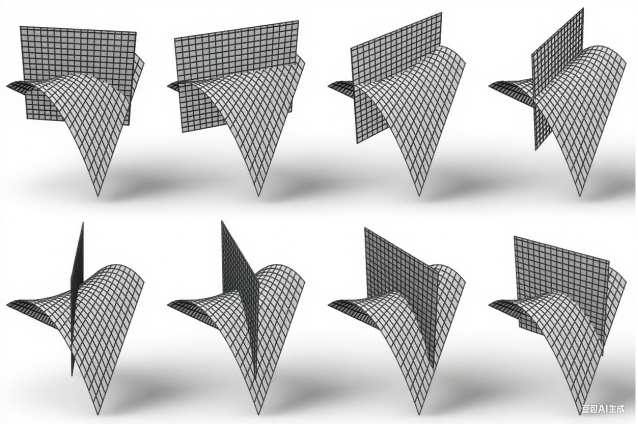
The Gauss map and the second fundamental form

Given a regular parametrized surface M, the function $\mathbf{n} : M \to \Sigma$ that assigns to each point $P \in M$ the unit normal $\mathbf{n}(P)$, is called the *Gauss map* of M.

Example

- On a plane, the tangent plane never changes, so the Gauss map is a constant.
- On a cylinder, the tangent plane is constant along the rulings, so the Gauss map sends the entire surface to an equator of the sphere.
- Ona sphere centered at the origin, the Gauss map is merely the position vector.

Intuitively, the information of the shape of M at certain point P is encoded in the curvature at P of various curves in M. We consider the normal slices of M, that is, we slice M with the plane through P spanned by $\mathbf{n}(P)$ and a unit vector $\mathbf{V} \in T_P M$.



Let $\boldsymbol{\alpha}$ be the arclength parametrized curve obtained by taking such a normal slice. We have $\boldsymbol{\alpha}(0) = P$ and $\boldsymbol{\alpha}'(0) = \mathbf{V}$. Obviously by definition of the slice, the curve lies in the plane spanned by $\mathbf{n}(P)$ and $\mathbf{V}$, the principal normal of the curve at P must be $\mathbf{n}(P)$. Since $(\mathbf{n} \circ \boldsymbol{\alpha}(s)) \cdot \mathbf{T}(s) = 0$, we have that

$$\kappa(P) = \kappa\mathbf{N} \cdot \mathbf{n}(P) = \mathbf{T}'(0) \cdot \mathbf{n}(P) = -\mathbf{T}'(0) \cdot (\mathbf{n} \circ \boldsymbol{\alpha})'(0) = -D_{\mathbf{V}}\mathbf{n}(P) \cdot \mathbf{V},$$

where the second $=$ comes from Frenet formula. A careful study of $D_{\mathbf{V}}\mathbf{n}(P)$ gives the following result.

Proposition

For any $\mathbf{V} \in T_P M$, the directional derivative $D_{\mathbf{V}}\mathbf{n}(P) \in T_P M$. Moreover, the linear map $S_P : T_P M \to T_P M$ defined by

$$S_P(\mathbf{V}) = -D_{\mathbf{V}}\mathbf{n}(P)$$

is a symmetric linear map, i.e., for any $\mathbf{U}, \mathbf{V} \in T_P M$, we have

$$S_P(\mathbf{U}) \cdot \mathbf{V} = \mathbf{U} \cdot S_P(\mathbf{V})$$

S_P is called the shape operator at P.

The second fundamental form is defined as follows. If $\mathbf{U}, \mathbf{V} \in T_P M$, we set

$$II_P(\mathbf{U}, \mathbf{V}) = S_P(\mathbf{U}) \cdot \mathbf{V}.$$

Covariant Derivative, Parallel transport, Geodesics

Definition

Let X be a differentiable vector field in an open set $U \subset M$ and $p \in U$. Let $\mathbf{y} \in T_pM$. Consider a parametrized curve

$$\boldsymbol{\alpha} : (-\epsilon, \epsilon) \to U,$$

with $\boldsymbol{\alpha}(0) = p$ and $\boldsymbol{\alpha}'(0) = \mathbf{y}$, and let $X(t)$, $t \in (-\epsilon, \epsilon)$, be the restriction of the vector field X to the curve $\boldsymbol{\alpha}$. The vector obtained by the normal projection of $\frac{dX}{dt}(0)$ onto the plane T_pM is called the covariant derivative at p of the vector field X relative to the vector $\mathbf{y}$. This covariant derivative is denoted by

$$\nabla_{\mathbf{y}} X(p) = \frac{dX}{dt} - (\frac{dX}{dt} \cdot \mathbf{n})\mathbf{n}.$$

Definition

A vector field X along a parametrized curve $\boldsymbol{\alpha} : I \to M$ is saied to be parallel if $\nabla_{\boldsymbol{\alpha}'} X = 0$ for all points in I.

Motivations

- Intuitively, in $\mathbb{R}^3$, two vectors $\mathbf{v}$ (starting from p) and $\mathbf{w}$ (starting from q) are parallel provided that we obtain $\mathbf{w}$ when we move $\mathbf{v}$ "parallel to itself" from p to q, i.e., if $\mathbf{w} = \mathbf{v}$.
- On a surface, how should we compare a tangent vector at one point of the surface to a tangent vector at another and determine if they are "parallel"?

Proposition

Let I be an interval in $\mathbb{R}$ with $0 \in I$. Given a curve $\boldsymbol{\alpha} : I \to M$ with $\boldsymbol{\alpha}(0) = p$ and $X_0 \in T_pM$, there is a unique parallel vector field X defined along $\boldsymbol{\alpha}$ with $X(p) = X_0$.

Given a curve and a tangent vector on the curve, there exists a unique way to move the vector along the curve in the sense of parallelism. And we can have the following notion of comparing two tangent vectors on a surface at different tangent spaces.

Definition

If $\boldsymbol{\alpha}$ is a path from p to q, we refer to $X(q)$ as the parallel translate of $X(p) = X_0 \in T_pM$ along $\boldsymbol{\alpha}$.

Definition

We say a parametrized curve in a surface M is a geodesic if its tangent vector is parallel along the curve, i.e., $\nabla_{\boldsymbol{\alpha}'}\boldsymbol{\alpha}' = 0$.

Proposition

Given a point $p \in M$ and $\mathbf{v} \in T_pM$, there exists $\epsilon > 0$ and a unique geodesic $\boldsymbol{\alpha} : (-\epsilon, \epsilon) \to M$ with $\boldsymbol{\alpha}(0) = p$ and $\boldsymbol{\alpha}'(0) = \mathbf{v}$.

Equation of movement of a forceless particle.

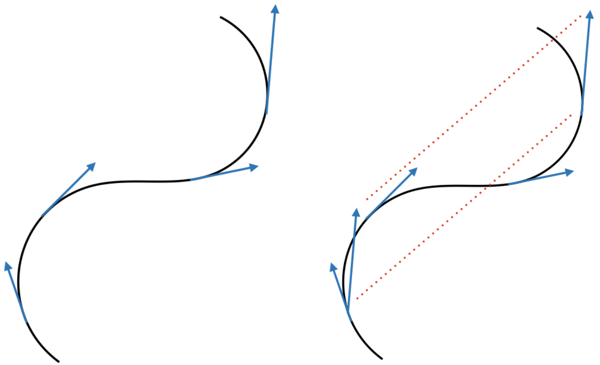
- Suppose we are trying to solve the trajectory of a forceless particle in $\mathbb{R}^3$ according to Newton's law of inertial;
- The equation that we are solving is the following

$$\boldsymbol{\alpha}''(t) = 0$$

  whose solution is $\boldsymbol{\alpha}(t) = t(c_1, c_2, c_3)^\top$;
- Question: How to measure the accelerate physically?

- To "measure" the accelerate, we expect to know a velocity field along the trajectory, say $\boldsymbol{\alpha}'(t)$, and compute the mean

$$\frac{\boldsymbol{\alpha}'(t+\Delta t)-\boldsymbol{\alpha}'(t)}{\Delta t}$$



- $\boldsymbol{\alpha}'(t+\Delta t)$ and $\boldsymbol{\alpha}'(t)$ live in different tangent spaces of $\mathbb{R}^2$, $\boldsymbol{\alpha}'(t+\Delta t) \in T_{\boldsymbol{\alpha}(t+\Delta t)}\mathbb{R}^2$ while $\boldsymbol{\alpha}'(t) \in T_{\boldsymbol{\alpha}(t)}\mathbb{R}^2$;
- A hidden step: move $\boldsymbol{\alpha}'(t+\Delta t)$ to the tangent space at $\boldsymbol{\alpha}(t)$ before computing $\boldsymbol{\alpha}'(t+\Delta t) - \boldsymbol{\alpha}'(t)$;
- To extend $\boldsymbol{\alpha}'(t+\Delta t)$ locally to a vector $\mathbf{v}(p_1)$ on $\boldsymbol{\alpha}(t)$ parallel to $\boldsymbol{\alpha}'(t+\Delta t)$, we just need a well defined metric (first fundamental form) in $T_{\boldsymbol{\alpha}(t+\Delta t)}\mathbb{R}^2$;
- The identification of different tangent spaces is a special feature of Euclidean geometry.

Since the covariant derivative (directional derivative) of a vector field X on a surface is defined to be

$$\nabla_{\mathbf{v}} X = (D_{\mathbf{v}} X)^{||} = D_{\mathbf{v}} X - (D_{\mathbf{v}} X \cdot \mathbf{n})\mathbf{n},$$

it is natural to talk about the change of X along a curve based on $\nabla_{\boldsymbol{\alpha}'(t)} X(\boldsymbol{\alpha}(t))$, recall that $\nabla_{\boldsymbol{\alpha}'(t)} X(\boldsymbol{\alpha}(t)) = \mathbf{0}$ is called *covariant constant* or *parallel*, which generalizes the parallelism in Euclidean space.

To prove that with a well defined covariant derivative, one can move a vector X along a curve $\boldsymbol{\alpha}(t)$ such that $\nabla_{\boldsymbol{\alpha}'(t)} X(\boldsymbol{\alpha}(t)) = \mathbf{0}$, we need Christoffel symbols. Given a parametrized surface $\mathbf{x} : U \to M$, we have

$$\begin{aligned}
\nabla_{\mathbf{x}_u}\mathbf{x}_u &= (\mathbf{x}_{uu})^{||} = \Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v \\
\nabla_{\mathbf{x}_v}\mathbf{x}_u &= (\mathbf{x}_{uv})^{||} = \Gamma^u_{uv}\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v = \nabla_{\mathbf{x}_u}\mathbf{x}_v \\
\nabla_{\mathbf{x}_v}\mathbf{x}_v &= (\mathbf{x}_{vv})^{||} = \Gamma^u_{vv}\mathbf{x}_u + \Gamma^v_{vv}\mathbf{x}_v.
\end{aligned}$$

$\Gamma^{\cdot}_{\cdot\cdot}$ are functions of (u, v).

Assuming $\boldsymbol{\alpha}$ lies in a parametrized $\mathbf{x} : U \to M$, set $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$ and write

$$X(\boldsymbol{\alpha}(t)) = a(t)\mathbf{x}_u(u(t), v(t)) + b(t)\mathbf{x}_v(u(t), v(t)).$$

Then

$$\boldsymbol{\alpha}'(t) = u'(t)\mathbf{x}_u + v'(t)\mathbf{x}_v.$$

So, by the product rule and chain rule, we have

$$\begin{aligned}\nabla_{\boldsymbol{\alpha}'(t)} X &= \left((X \circ \boldsymbol{\alpha})'(t)\right)^{||} = \left(\frac{d}{dt}\left(a(t)\mathbf{x}_u(u(t), v(t)) + b(t)\mathbf{x}_v(u(t), v(t))\right)\right)^{||} \\ &= a'(t)\mathbf{x}_u + b'(t)\mathbf{x}_v + a(t)\left(\frac{d}{dt}\mathbf{x}_u(u(t), v(t))\right)^{||} + b(t)\left(\frac{d}{dt}\mathbf{x}_v(u(t), v(t))\right)^{||}\end{aligned}$$

$$
\begin{aligned}
&= a'(t)\mathbf{x}_u + b'(t)\mathbf{x}_v + a(t)\left(u'(t)\mathbf{x}_{uu} + v'(t)\mathbf{x}_{uv}\right)^{||} + b(t)\left(u'(t)\mathbf{x}_{vu} + v'(t)\mathbf{x}_{vv}\right)^{||} \\
&= a'(t)\mathbf{x}_u + b'(t)\mathbf{x}_v + a(t)\left(u'(t)(\Gamma^u_{uu}\mathbf{x}_u + \Gamma^v_{uu}\mathbf{x}_v) + v'(t)(\Gamma^u_{uv})\mathbf{x}_u + \Gamma^v_{uv}\mathbf{x}_v\right) \\
&+ b(t)\left(u'(t)(\Gamma^u_{vu}\mathbf{x}_u + \Gamma^v_{vu}\mathbf{x}_v) + v'(t)(\Gamma^u_{vv}\mathbf{x}_u + \Gamma^v_{vv}\mathbf{x}_v)\right) \\
&= \left(a'(t) + a(t)(\Gamma^u_{uu}u'(t) + \Gamma^u_{uv}v'(t)) + b(t)(\Gamma^u_{vu}u'(t) + \Gamma^u_{vv}v'(t))\right)\mathbf{x}_u \\
&+ \left(b'(t) + a(t)(\Gamma^v_{uu}u'(t) + \Gamma^v_{uv}v'(t)) + b(t)(\Gamma^v_{vu}u'(t) + \Gamma^v_{vv}v'(t))\right)\mathbf{x}_v
\end{aligned}
$$

So to say X is parallel along the curve $\boldsymbol{\alpha}$ is to say that $a(t)$ and $b(t)$ are solutions of the linear system of first order ODE

$$
\begin{aligned}
a'(t) + a(t)(\Gamma^u_{uu}u'(t) + \Gamma^u_{uv}v'(t)) + b(t)(\Gamma^u_{vu}u'(t) + \Gamma^u_{vv}v'(t)) = 0 \\
b'(t) + a(t)(\Gamma^v_{uu}u'(t) + \Gamma^v_{uv}v'(t)) + b(t)(\Gamma^v_{vu}u'(t) + \Gamma^v_{vv}v'(t)) = 0
\end{aligned}
$$

Fundamental theorem of ODE implies the existence and uniqueness of the solution.

Example

$$\mathbf{x}(u,v) = (\sin u \cos v, \sin u \sin v, \cos u), \quad 0 < u < \pi, 0 \le v \le 2\pi.$$

and the Christoffel symbols of sphere are

$$\begin{bmatrix} \Gamma^u_{uu} & \Gamma^u_{uv} & \Gamma^u_{vv} \\ \Gamma^v_{uu} & \Gamma^v_{uv} & \Gamma^v_{vv} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\sin u \cos u \\ 0 & \cot u & 0 \end{bmatrix}$$

Fix a latitude circle $u = u_0$ $(u_0 \neq 0, \pi)$ on the unit sphere. Compute the parallel transport of the vector $\mathbf{x}_v$ starting at the point p given by $u = u_0$, $v = 0$, once around the circle, counterclockwise. Parametrize the curve by $u(t) = u_0$, $v(t) = t$, $t \le t \le 2\pi$. Then the parallel transport equation in parameter space (u, v) is

$$a'(t) = \sin u_0 \cos u_0 b(t), \quad a(0) = 0$$

$$b'(t) = -\cot u_0 a(t), \quad b(0) = 1$$

Geodesic in parameter space

The equation $\nabla_{\boldsymbol{\alpha}'}\boldsymbol{\alpha}'$ can be solved in parameters. Suppose the equation for the curve $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$ to be a geodesic. Since

$$X = \boldsymbol{\alpha}'(t) = u'(t)\mathbf{x}_u + v'(t)\mathbf{x}_v$$

we have $a(t) = u'(t)$ and $b(t) = v'(t)$, and the resulting equations are

$$u''(t) + \Gamma^u_{uu}u'(t)^2 + 2\Gamma^u_{uv}u'(t)v'(t) + \Gamma^u_{vv}v'(t)^2 = 0$$
$$v''(t) + \Gamma^v_{uu}u'(t)^2 + 2\Gamma^v_{uv}u'(t)v'(t) + \Gamma^v_{vv}v'(t)^2 = 0$$

Again, fundamental theorem of ODE guarantees the existence and uniqueness of the curve $\boldsymbol{\alpha} : (-\epsilon, \epsilon) \to M$ with $\boldsymbol{\alpha}(0) = p$ and $\boldsymbol{\alpha}'(0) = \mathbf{v}$ and satisfying $\nabla_{\boldsymbol{\alpha}'(t)}\boldsymbol{\alpha}'(t) = 0$ for all $t \in (-\epsilon, \epsilon)$.

Example

- Let $\mathbf{x}(u,v) = (u,v)$ be a parametrization of the plane. Then all the Christoffel symbols vanish and the geodesics are the solutions of

$$u''(t) = v''(t) = 0.$$

Gauss-Bonnet Theorem

Definition

Let $\boldsymbol{\alpha}$ be a closed curve in a surface M. The angle through which a vector turns *relative to the given framing* as we parallel translate it once around the curve $\boldsymbol{\alpha}$ is called the *holonomy* around $\boldsymbol{\alpha}$

We will work in an orthogonal parametrization and define a framing by setting

$$\mathbf{e}_1 = \frac{\mathbf{x}_u}{\sqrt{E}} \quad \text{and} \quad \mathbf{e}_2 = \frac{\mathbf{x}_v}{\sqrt{G}}.$$

Since $\mathbf{e}_1$ and $\mathbf{e}_2$ give an orthonormal basis for the tangent space, all the intrinsic curvature information is encapsulated in knowing how $\mathbf{e}_1$ twists towards $\mathbf{e}_2$ as we move around the surface. In particular, if

$$\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t)), \quad a \leq t \leq b,$$

we can set the infinitesimal change of angle

$$\phi_{12}(t) = \frac{d}{dt}\left(\mathbf{e}_1(u(t), v(t))\right) \cdot \mathbf{e}_2(u(t), v(t)).$$

Or we can also write

$$\phi_{12} = \nabla_{\boldsymbol{\alpha}'}\mathbf{e}_1 \cdot \mathbf{e}_2.$$

The explicit formula for ϕ_{12} can be derived as follows.

Proposition

In an orthogonal parametrization with $\mathbf{e}_1$ *and* $\mathbf{e}_2$*, we have*

$$\phi_{12} = \frac{1}{2\sqrt{EG}}(-E_v u' + G_u v').$$

Proof.

$$\phi_{12} = \frac{d}{dt}\left(\frac{\mathbf{x}_u}{\sqrt{E}}\right) \cdot \frac{\mathbf{x}_v}{\sqrt{G}} = \frac{1}{\sqrt{EG}}(\mathbf{x}_{uu}u' + \mathbf{x}_{uv}v') \cdot \mathbf{x}_v$$

Since the term that arise from differentiating $\sqrt{E}$ will involve $\mathbf{x}_u \cdot \mathbf{x}_v = 0$, the result holds.

□

Proposition

The holonomy around the closed curve C equals $\Delta\psi = -\int_a^b \phi_{12}(t)dt$.

Proof.

Suppose that $\boldsymbol{\alpha}$ is a closed curve and we are investigating the holonomy around $\boldsymbol{\alpha}$. If $\mathbf{e}_1$ happens to be parallel along $\boldsymbol{\alpha}$, then the holonomy will be 0. If not, consider $X(t)$ to be the parallel translation of $\mathbf{e}_1$ along $\boldsymbol{\alpha}(t)$ and write

$$X(t) = \cos\psi(t)\mathbf{e}_1 + \sin\psi(t)\mathbf{e}_2$$

Then X is parallel along $\boldsymbol{\alpha}$ if and only if

$$\mathbf{0} = \nabla_{\boldsymbol{\alpha}'}X = \nabla_{\boldsymbol{\alpha}'}(\cos\psi\mathbf{e}_1 + \sin\psi\mathbf{e}_2) = (\phi_{12} + \psi')(-\sin\psi\mathbf{e}_1 + \cos\psi\mathbf{e}_2).$$

Thus X is parallel if and only if $\psi'(t) = -\phi_{12}(t)$. And therefore, we have

$$\Delta\psi = -\int_a^b \phi_{12}(t)dt.$$

Suppose now that $\boldsymbol{\alpha}$ is an arclength-parametrized curve and write

$$\boldsymbol{\alpha}(s) = \mathbf{x}(u(s), v(s))$$

and

$$\mathbf{T}(s) = \boldsymbol{\alpha}'(s) = \cos\theta(s)\mathbf{e}_1 + \sin\theta(s)\mathbf{e}_2.$$

We have the following

Proposition

When $\boldsymbol{\alpha}$ is an arclength-parametrized curve, the geodesic curvature of $\boldsymbol{\alpha}$ is given by

$$\kappa_g(s) = \phi_{12}(s) + \theta'(s)$$

Corollary

When R is a region with smooth boundary and lying in an orthogonal parametrization, the holonomy around ∂R is $\Delta\psi = \int_R K dA$.

Proof.

$$\int_0^L \phi_{12}(s)ds = \int_0^L \frac{1}{2\sqrt{EG}}(-E_v u'(s) + G_u v'(s))ds = \int_{\partial R} \frac{1}{2\sqrt{EG}}(-E_v du + G_u dv)$$

$$= \iint_R \left(\left(\frac{E_v}{2\sqrt{EG}} \right)_v + \left(\frac{G_u}{2\sqrt{EG}} \right)_u \right) du dv$$

$$= \iint_R \frac{1}{2\sqrt{EG}} \left(\left(\frac{E_v}{\sqrt{EG}} \right)_v + \left(\frac{G_u}{\sqrt{EG}} \right)_u \right) \underbrace{\sqrt{EG} du dv}_{dA}$$

$$= -\iint_R K dA$$

□

Note that

$$\int_{\partial R} \kappa_g ds = \int_{\partial R} \phi_{12} ds + \theta(L) - \theta(0)$$

so the total angle through which the tangent vector to ∂R turns is given by

$$\Delta\theta = \int_{\partial R} \kappa_g ds + \int_R K dA.$$

In particular, when R is simply connected, $\Delta\theta = 2\pi$.

Theorem (Local Gauss-Bonnet)

Suppose R is a simply connected region with piecewise smooth boundary and lying in an orthogonal parametrization. If $C = \partial R$ has exterior angles $\epsilon_j, j = 1, ..., \ell$, then

$$\int_{\partial R} \kappa_g ds + \int_R K dA + \sum_{j=1}^{\ell} \epsilon_j = 2\pi.$$

Questions?