Riemannian Geometry

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Riemannian Metric

A Riemannian metric is an inner product smoothly defined on each tangent space to a manifold. An inner product on a vector space V is a function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ that has the following properties:

1. bilinearity: $\langle a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2, \mathbf{v} \rangle = a_1 \langle \mathbf{u}_1, \mathbf{v} \rangle + a_2 \langle \mathbf{u}_2, \mathbf{v} \rangle$ and $\langle \mathbf{v}, a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 \rangle = a_1 \langle \mathbf{v}, \mathbf{u}_1 \rangle + a_2 \langle \mathbf{v}, \mathbf{u}_2 \rangle$;
2. symmetry: $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$;
3. positive definiteness: if $\mathbf{u} \neq 0$ then $\langle \mathbf{u}, \mathbf{u} \rangle > 0$.

Definition

A Riemannian metric on a smooth manifold M is a mapping g that assigns to every point $p \in M$ an inner product $g_p = \langle \cdot, \cdot \rangle_p$ on the tangent space T_pM, which depends smoothly on $p \in M$, in the sense that, for any two smooth vector fields X and Y on M, the function

$$p \in M \to \langle X_p, Y_p \rangle_p \in \mathbb{R}$$

is smooth. A manifold M endowed with a Riemannian metric is called a Riemannian manifold.

Geometric quantities defined by Riemannian metric

- The length of a tangent vector $\mathbf{v} \in T_p M$ is defined by

$$\|\mathbf{v}\| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2};$$

- For a given curve $c : [a, b] \to M$, the arc-length of c between a and b is

$$L(c) = \int_a^b \left\| \frac{dc}{dt}(t) \right\| dt;$$

- For a pair of regular curves c_1 and c_2 that meet at a point $p = c_1(t_0) = c_2(t_0)$, the angle between the two curves at p is given by

$$\cos\theta = \frac{\langle \frac{dc_1}{dt}(t_0), \frac{dc_2}{dt}(t_0) \rangle}{\left\| \frac{dc_1}{dt}(t_0) \right\| \left\| \frac{dc_2}{dt}(t_0) \right\|}$$

Connection

Definition (Connection)

A connection on a manifold M is a smooth map that assigns to every pair of smooth vector fields X and Y on M another smooth vector field $\nabla_X Y$ on M, satisfying the following properties:

- bilinearity:$\nabla_{a_1 X_1 + a_2 X_2} Y = a_1 \nabla_{X_1} Y + a_2 \nabla_{X_2} Y$ and $\nabla_X(a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2$, for any $a_1, a_2 \in \mathbb{R}$ and any smooth vector fields X_1, X_2, Y_1, Y_2 on M;
- linearity in X over the smooth functions: $\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y$;
- product rule in Y: $\nabla_X(fY) = f \nabla_X Y + X(f) Y$.

Example

$\nabla_{\mathbf{v}} X$ defined on parametrized surface is a connection.

Connection in local coordinates

Since $\nabla_{\partial/\partial x_i}\partial/\partial x_j$ is a vector field, its value at each point can be expressed as a linear combination of the tangent space vector basis

$$\nabla_{\frac{\partial}{\partial x_i}}\frac{\partial}{\partial x_j}=\sum_{k=1}^{m}\Gamma_{ij}^{k}\frac{\partial}{\partial x_k}.$$

The n^3 smooth function Γ_{ij}^{k} uniquely determine the connection. Indeed, if

$$X=\sum_{i=1}^{m}u_i\frac{\partial}{\partial x_i}\quad Y=\sum_{i=1}^{m}v_i\frac{\partial}{\partial x_i}$$

are two smooth vector fields, then

$$\nabla_X Y=\sum_{k=1}^{m}\left(\sum_{i,j=1}^{m}u_iv_j\Gamma_{ij}^{k}+\sum_{i=1}^{m}u_i\frac{\partial v_k}{\partial x_i}\right)\frac{\partial}{\partial x_k}\quad \text{(check by definition)}$$

Proposition (Covariant derivative along a curve induced by connection)

Let M be a differentiable manifold with an connection ∇. There exists a unique correspondence which associates to a vector field V along the differentiable curve $c : I \to M$ another vector field $\frac{DV}{dt}$ along c, called the covariant derivative of V along c, such that

a) $\frac{D}{dt}(V+W) = \frac{DV}{dt} + \frac{DW}{dt}$;

b) $\frac{D}{dt}(fV) = \frac{df}{dt}V + f\frac{DV}{dt}$, *where W is a vector field along c and f is a differentiable function on I;*

c) *If V is induced by a vector field $Y \in \mathcal{X}(M)$, i.e, $V(t) = Y(c(t))$, then* $\frac{DY}{dt} = \nabla_{c'(t)}Y$.

proof

In local coordinate, we can define $\frac{DV}{dt} = \sum_j \frac{dv_j}{dt} X_j + \sum_{i,j} \frac{dx_i}{dt} v_j \nabla_{X_i} X_j$, then the desired properties can be verified by properties of $\nabla_{X_i} X_j$. To show uniqueness, we suppose that there exists a correspondence satisfying a)-c). Let $(x_1(t), ..., x_n(t))$ be the local expression of $c(t)$, and the local expression of V is $\sum_j v_j X_j$. By a) and b), we have

$$\frac{DV}{dt} = \sum_j \frac{dv_j}{dt} X_j + \sum_j v_j \frac{DX_j}{dt}.$$

By c) and definition of connection,

$$\frac{DX_j}{dt} = \nabla_{c'(t)} X_j = \nabla_{\sum \frac{dx_i}{dt} X_i} X_j = \sum_i \frac{dx_i}{dt} \nabla_{X_i} X_j.$$

Therefore,

$$\frac{DV}{dt} = \sum_j \frac{dv_j}{dt} X_j + \sum_{i,j} \frac{dx_i}{dt} v_j \nabla_{X_i} X_j$$

which is uniquely determined by $c(t)$, V and bases of coordinates X_i.

Proposition

Given an connection ∇ *on* M*, a curve* $\boldsymbol{\alpha} : I \to M$*, and a tangent vector* $X_{\boldsymbol{\alpha}(t_0)}$ *at a point* $\boldsymbol{\alpha}(t_0)$ *on the curve, there exists a unique parallel vector field* $X_{\boldsymbol{\alpha}(t)}$ *along* $\boldsymbol{\alpha}$ *that extends* $X_{\boldsymbol{\alpha}(t_0)}$

Proof.

Let $(x_1, ..., x_m)$ be a local coordinate system near $\boldsymbol{\alpha}(t_0)$ and let

$$X_{\boldsymbol{\alpha}(t_0)} = \sum_{k=1}^{m} u_k \frac{\partial}{\partial x_k}$$

be the representation of $X_{\boldsymbol{\alpha}(t_0)}$ in these coordinates. The condition $\nabla_{\boldsymbol{\alpha}'(t)} X = 0$ applies and it induces differential equations

$$\begin{cases} \frac{dv_k}{dt} + \sum_{i,j=1}^{m} v_j \frac{dx_i}{dt} \Gamma_{ij}^k = 0 \\ v_k(t_0) = u_k \end{cases}$$

cont. proof

$\frac{dx_i}{dt}$ is known if the curve $\boldsymbol{\alpha}(t)$ is given, thus the differential equations have unique solution locally by fundamental theorem of ODEs.

As what is defined for parallel transport on surfaces, the covariant derivative also defines a parallel transport for general Riemannian manifold.

Definition

Let $\boldsymbol{\alpha} : I \to M$ be a curve in M and $\boldsymbol{\alpha}(t_0)$ be a point on the curve. The mapping $P_{\boldsymbol{\alpha}(t_0)}^{\boldsymbol{\alpha}(t)} : T_{\boldsymbol{\alpha}(t_0)}M \to T_{\boldsymbol{\alpha}(t)}M$ defined by $P_{\boldsymbol{\alpha}(t_0)}^{\boldsymbol{\alpha}(t)} V_{\boldsymbol{\alpha}(t_0)} = V_{\boldsymbol{\alpha}(t)}$, where $V_{\boldsymbol{\alpha}(t_0)} \in T_{\boldsymbol{\alpha}(t_0)}M$, and $V_{\boldsymbol{\alpha}(t)}$ is the unique parallel extension of $V_{\boldsymbol{\alpha}(t_0)}$ along $\boldsymbol{\alpha}(t)$. This mapping is called parallel transport.

Riemannian connection

Definition (Compatibility)

Let M be a manifold with a connection ∇ and Riemannian metric $\langle\cdot,\cdot\rangle$. A connection is said to be compatible with the metric $\langle\cdot,\cdot\rangle$, when for any smooth curve c and any pair of parallel vector fields P and P' along c, we have $\langle P, P'\rangle = \text{constant}$.

Proposition

Let M be a Riemannian manifold. A connection ∇ on M is compatible with a metric if and only if for any vector fields V and W along the differentiable curve $c : I \to M$ we have

$$d\langle V, W\rangle = \langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle.$$

Proof.

($\Rightarrow$) Choose an orthonormal basis $\{P_1(t_0), ..., P_n(t_0)\}$ of $T_{x(t_0)}M$. Using previous proposition on parallel transport of vector fields along curve, we can extend $P_i(t_0)$ along c by parallel transport. Since ∇ is compatible with the metric, the resulted fields $\{P_1(t), ..., P_n(t)\}$ is an orthonormal basis of $T_{c(t)}M$, for all $t \in I$. Write

$$V = \sum_i v_i P_i \quad W = \sum_i w_i P_i,$$

it follows from definition of D/dt that

$$\frac{DV}{dt} = \sum_i \frac{dv_i}{dt} P_i, \quad \frac{DW}{dt} = \sum_i \frac{dw_i}{dt} P_i.$$

□

Therefore,

$$\langle \frac{DV}{dt}, W\rangle + \langle V, \frac{DW}{dt}\rangle = \sum_i \left(\frac{dv_i}{dt} w_i + \frac{dw_i}{dt} v_i \right) = \frac{d}{dt} \left(\sum_i v_i w_i \right) = \frac{d}{dt}\langle V, W\rangle.$$

($\Leftarrow$) Trivial by the equation $\frac{d}{dt}\langle V, W\rangle = \langle \frac{DV}{dt}, w\rangle + \langle V, \frac{DW}{dt}\rangle$.

Definition

An connection ∇ on a smooth manifold M is said to be *symmetric* when

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

Theorem (Levi-Civita)

Given a Riemannian manifold M, there exists a unique connection ∇ on M satisfying

a) *∇ is symmetric;*

b) *∇ is compatible with the Riemannian metric.*

Remark

In many contexts, authors refer to Levi-Civita (Riemannian) connection if no specification provided. In local coordinates, $\nabla_{X_i} X_j = \sum \Gamma_{ij}^k X_k$, where

$$\Gamma_{ij}^m = \frac{1}{2} \sum_k \left(\frac{\partial g_{jk}}{\partial x_i} + \frac{\partial g_{ki}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_k} \right) g^{km}$$

Geodesics in local coordinates

Definition

A smooth curve $\gamma : I \to M$ is called a geodesic if its velocity vector field is parallel, i.e., $\nabla_{\gamma'}\gamma' = 0$ which agrees with the geodesics defined on surfaces.

We can rewrite equation $\nabla_{\gamma'}\gamma' = 0$ with respect to a local coordinate system $(x_1, ..., x_m)$. If $(x_1(t), ..., x_m(t))$ represents a geodesic, then using the following covariant derivative equation of a vector field along a curve,

$$\sum_{k=1}^{m}\left(\frac{dv_k}{dt} + \sum_{i,j} v_j \frac{dx_i}{dt}\Gamma_{ij}^{k}\right)\frac{\partial}{\partial x_k}$$

we obtain the second order system

$$\frac{d^2 x_k}{dt^2} + \sum_{i,j=1}^{m} \frac{dx_i}{dt}\frac{dx_j}{dt}\Gamma_{ij}^{k} = 0,$$

whose solution is known by fundamental theorem of ODEs.

Remark

Before using fundamental theorem of ODEs, we need to change the second oder ode to first oder by introducing new variables

$$y_i = \frac{dx_i}{dt}$$

and this reduces the geodesic equation to

$$\begin{cases} \frac{dx_i}{dt} & = y_i \\ \frac{dy_i}{dt} & = -\sum_{i,j=1}^{m} \Gamma_{ij}^k y_i y_j \end{cases}$$

Integral of this first order system is called geodesic flow on the tangent bundle TM.

Example (Geodesics on hyperbolic plane)

We show that the geodesics of the Poincaré half plane are vertical lines and semi-circles with the center on the x-axis. For the hyperbolic plane, we have $g_{11} = g_{22} = \frac{1}{y^2}$, and $g_{12} = g_{21} = 0$. The entries of the inverse matrix g^{kl} are $g^{11} = g^{22} = y^2$, and $g^{12} = g^{21} = 0$. We can also compute Christoffel symbols

$$\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\Gamma^2_{11} = -\frac{1}{y},$$

while the rest equal 0. The geodesic equations are

$$\begin{cases} \frac{d^2x}{dt^2} - \frac{2}{y}\frac{dx}{dt}\frac{dy}{dt} = 0 \\ \frac{d^2y}{dt^2} + \frac{1}{y}\left(\frac{dx}{dt}\right)^2 - \frac{1}{y}\left(\frac{dy}{dt}\right)^2 = 0 \end{cases}$$

- If $\frac{dx}{dt} = 0$ for all t, the first equation is satisfied, so the vertical lines $x =$ constant are geodesics.
- If $\frac{dx}{dt} \neq 0$ at some t, then we can locally solve for x as a function of y. If $u = \frac{dx}{dy}$ then $\frac{dx}{dt} = \frac{udy}{dt}$ and so, by the chain rule,

$$\frac{d^2x}{dt^2} = \frac{du}{dy}\left(\frac{dy}{dt}\right)^2 + u\frac{d^2y}{dt^2}.$$

By substituting $\frac{d^2x}{dt^2}$ from the first equation and $\frac{d^2y}{dt^2}$ from the second equation, after simplification we obtain

$$\frac{du}{dy} = \frac{u^3 + u}{y}.$$

Integration by partial fraction gives

$$u(y) = \frac{dx}{dy} = \pm\frac{cy}{\sqrt{1 - c^2y^2}}$$

so

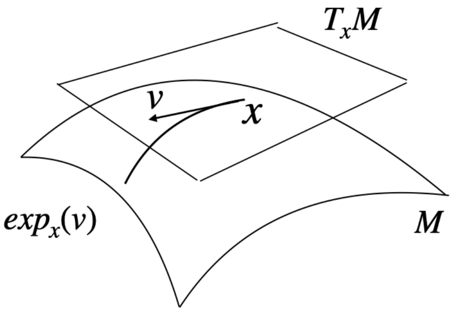
$$x = \pm \int \frac{cy}{\sqrt{1 - c^2y^2}} dy = \mp \sqrt{\left(\frac{1}{c}\right)^2 - y^2} + d$$

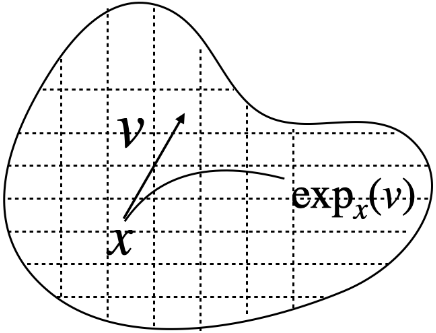
for some $d \in \mathbb{R}$.

The Exponential Map

The exponential map describes the dependence of geodesics emanating from the same point on their initial velocity. If $v \in T_pM$ and the geodesic $\gamma_{p,v}(t)$ is defined on $[0, 1]$, then we define $\mathrm{Exp}_p(v)$ by

$$\mathrm{Exp}_p(v) = \gamma_{p,v}(1). \quad \text{(why it's well defined?)}$$





Questions?