Differentiable Manifolds and Mappings

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Review of Topology

A mapping $f : U \subset \mathbb{R}^m \to \mathbb{R}^n$ is continuous at a point $x_0 \in U$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that, for each $x \in U$

$$\|f(x) - f(x_0)\| < \epsilon \quad \text{provided} \quad \|x - x_0\| < \delta.$$

A mapping is said to be continuous if it is continuous at every point of its domain. A sufficient condition for a mapping defined on an open set U in $\mathbb{R}^m$ to be continuous is that f is differentiable at every point $x_0 \in U$.

Topological space

A topological space is a set X together with a collection $\mathcal{T}$ of subsets of X satisfying the following properties:

- The empty set $\emptyset$ and the total space X are in $\mathcal{T}$;
- The union of any collection of sets in $\mathcal{T}$ is a set in $\mathcal{T}$;
- The intersection of any finite collection of sets in $\mathcal{T}$ is a set in $\mathcal{T}$.

The sets in $\mathcal{T}$ are called **open sets** of the topological space. The collection $\mathcal{T}$ is called a **topology** on X.

Example

- The Euclidean space $\mathbb{R}^m$ with the open sets defined by Euclidean norm;
- If X is any set, the collection of all subsets of X is a topology on X, this is called the discrete topology;
- A distance function on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying the following properties for all $p, q, r \in X$:

1. $d(p, q) \geq 0$ and $d(p, q) = 0$ if and only if $p = q$;
2. $d(p, q) = d(q, p)$;
3. $d(p, q) \leq d(p, r) + d(r, q)$.

Submanifold of $\mathbb{R}^{n+k}$

The **unit n-sphere** is

$$S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\},$$

where $\|x\| = \left(\sum_{i=1}^{n+1} x_i^2\right)^{\frac{1}{2}}$. We introduce local coordinates in S^n as follows. For $j = 1, ..., n+1$ define open hemispheres

$$U_{2j-1} = \{x \in S^n : x_j > 0\},$$
$$U_{2j} = \{x \in S^n : x_j < 0\}.$$

For $i = 1, ..., 2n + 2$ define maps

$$\varphi_i : U_i \to \mathbb{R}^n,$$

$$\varphi_i(x) = (x_1, ..., \hat{x}_j, ..., x_{n+1})$$

if $i = 2j - 1$ or $2j$; this means the n-tuple obtained from x by deleting the jth coordinate.
Each (φ_i, U_i) is called a "chart" for S^n; the set of all (φ_i, U_i) is an "atlas". In terms of this atlas, we say a map

$$f : S^n \to \mathbb{R}^k$$

is **differentiable of class** C^r in case each composite map

$$f \circ \varphi_i^{-1} : B \to \mathbb{R}^k$$

is C^r.

Manifolds in $\mathbb{R}^n$

Manifolds form one of the most important classes of spaces in mathematics. They are useful in the fields as differential geometry, theoretical physics, and optimization. To begin with, we will strict ourselves to manifolds that are submanifolds of Euclidean space $\mathbb{R}^n$. It is natural to consider submanifold since the first definition of manifold defined by Poincaré is actually submanifolds of Euclidean spaces. Formally, the definition of a manifold in $\mathbb{R}^n$ is given as follows.

Definition

Let $k > 0$. Suppose that M is a subspace of $\mathbb{R}^n$ having the following property: For each $p \in M$, there is a set V containing p that is open in M, a set U that is open in $\mathbb{R}^k$, and a continuous map $\varphi : U \to V$ carrying U onto V in a one-to-one fashion. In particular, φ is of class C^r, φ^{-1} is continuous, and $D\varphi(x)$ has rank k for each $x \in U$. Then M is called a k-manifold in $\mathbb{R}^n$ and the map φ is called a local coordinate system or a patch on M at p.

Derivatives and Tangents

Suppose that f is a smooth map of an open set in $\mathbb{R}^n$ into $\mathbb{R}^m$ and $\mathbf{x}$ is any point in its domain. Then for any vector $\mathbf{h} \in \mathbb{R}^n$, the derivative of f in the direction $\mathbf{h}$ is defined by the limit

$$df_{\mathbf{x}}[\mathbf{h}] = \lim_{t \to 0} \frac{f(\mathbf{x} + t\mathbf{h}) - f(\mathbf{x})}{t}.$$

With $\mathbf{x}$ fixed, we define a mapping $df_{\mathbf{x}}$ by assigning to each vector $\mathbf{h} \in \mathbb{R}^n$ the directional derivative $df_{\mathbf{x}}[\mathbf{h}] \in \mathbb{R}^m$, and this map is called the derivative of f at $\mathbf{x}$.

Differential Structures

Definition

A topological space M is called an n-dimensional manifold if it is locally homeomorphic to $\mathbb{R}^n$. That is, there is an open cover $\mathcal{U} = \{U_i\}_{i \in \Lambda}$ of M such that for each $i \in \Lambda$ there is a map

$$\varphi_i : U_i \to \mathbb{R}^n$$

which maps U_i homeomorphically onto an open subset of $\mathbb{R}^n$. We call (φ_i, U_i) a *chart or coordinate system* with domain U_i. The set of charts $\Phi = \{\varphi_i, U_i\}_{i \in \Lambda}$ is an *atlas*.

Two charts (φ_i, U_i) and (φ_j, U_j) are said to have C^r *overlap* if the *coordinate change*

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is of differentiability of class C^r, and $\varphi_i \circ \varphi_j^{-1}$ is also C^r. Here r can be a natural number, ∞, or ω (for real analytic).

An atlas Φ on M is called C^r if every pair of its charts has C^r overlap. In this case there is a unique maximal C^r atlas Ψ which contains Φ. A maximal C^r atlas on M is a C^r *differential structure*. M is called a *manifold of class* C^r.

Differential structures on submanifolds

Some manifolds are contained in other manifolds in a natural way, like $S^n \subset \mathbb{R}^{n+1}$. A subset A of a C^r manifold (M, Φ) is a C^r submanifold of (M, Φ) if for some integer $k \geq 0$, each point of A belongs to the domain of a chart $(\varphi, U) \in \Phi$ such that

$$U \cap A = \varphi^{-1}(\mathbb{R}^k)$$

where $\mathbb{R}^k \subset \mathbb{R}^n$ is the set of vectors whose last $n-k$ coordinates are 0. (φ, U) is called a submanifold chart for (M, A).

Differentiable Maps and the Tangent Bundle

Let M and N be C^r manifolds and $f : M \to N$ a map. A pair of charts (φ, U) for M and (ϕ, V) for N is *adapted to* f if

$$f(U) \subset V.$$

In this case the map

$$\phi \circ f \circ \varphi^{-1} : \varphi(U) \to \phi(V)$$

is defined; we call it the *local representation* of f in the given charts, *at the point* x if $x \in U$.

Let M be a C^{r+1} manifold with local charts $\{\varphi_i, U_i\}_{i \in I}$. A **tangent vector** to M is an equivalence class $[x, i, \mathbf{a}]$ of triples

$$(x, i, \mathbf{a}) \in M \times I \times \mathbb{R}^n$$

under the equivalence relation

$$[x, i, \mathbf{a}] = [y, j, \mathbf{b}]$$

if and only if $x = y$ and

$$D(\varphi_j \circ \varphi_i^{-1})(\varphi_i(x))\mathbf{a} = \mathbf{b}.$$

In other words, the derivative of the coordinate change at $\varphi_i(x)$ sends $\mathbf{a}$ to $\mathbf{b}$.

The set of all tangent vectors is TM, called the **tangent bundle** of M. The map

$$p = p_M : TM \to M,$$

$$[x, i, \mathbf{a}] \mapsto x$$

is well defined.
For any chart (φ_i, U_i) there is a well defined bijective map

$$T\varphi_i : TU_i \to \varphi_i(U_i) \times \mathbb{R}^n \subset \mathbb{R}^n \times \mathbb{R}^n$$
$$[x, i, \mathbf{a}] \mapsto (\varphi_i(x), \mathbf{a}).$$

And the change of coordinates map

$$(T\varphi_j) \circ (T\varphi_i)^{-1} : \varphi_i(U_i \cap U_j) \times \mathbb{R}^n \to \varphi_j(U_i \times U_j) \times \mathbb{R}^n$$
$$(y, \mathbf{a}) \mapsto (\varphi_j \circ \varphi_i^{-1}(y), D(\varphi_j \circ \varphi_i^{-1})(y)\mathbf{a}).$$

In one word, TM has a structure of manifold.

Tangent vectors as derivations

We can compute the directional derivative of a function in the direction of a tangent vector. The derivative of a function f in the direction of a vector $\mathbf{v}$ at a point p is given by

$$\frac{d}{dt} f(p + t\mathbf{v})|_{t=0}$$

equal to

$$\frac{df}{d\mathbf{v}} = \frac{d}{dt} f(c(t))|_{t=0}$$

for any smooth curve c with $c(0) = p$ and $c'(0) = \mathbf{v}$.

This idea carries over to a manifold M. Given a smooth mapping $f : M \to \mathbb{R}$, a point p on M, and a tangent vector $\mathbf{v}$ to M at p. From the definition of tangent vectors based on equivalence classes, this means that pick a local chart and a representative element $\mathbf{v} \in \mathbb{R}^n$.
Let c be one of the curves with $c(0) = p$ that represents $\mathbf{v}$. The derivative of f in direction of $\mathbf{v}$ is defined by

$$\mathbf{v}(f)(p) = \frac{d}{dt} f(c(t))|_{t=0}.$$

- With this definition, the partial derivative notation $\frac{\partial}{\partial x_i}$ is an element of the tangent space T_pM.
- Given a vector $\mathbf{v} = (v_1, ..., v_n)$ in local coordinates, the action of $\mathbf{v}$ on a function f, which is the directional derivative along $\mathbf{v}$, can be written as

$$\mathbf{v}(f) = \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x_i},$$

and this indicates that $\frac{\partial f}{\partial x_i}$ forms a basis of T_pM.

Note that the directional derivative $\mathbf{v}(f)$ behaves like the usual derivative operator, meaning that it is linear and satisfies the product rule:

- $\mathbf{v}(af + bg) = a\mathbf{v}(f) + b\mathbf{v}(g)$;
- $\mathbf{v}(fg) = f(p)\mathbf{v}(g) + g(p)\mathbf{v}(f)$.

This notion enables us to give another definition of tangent vectors:

Definition

Let $C^\infty(M)$ be the set of all C^∞ functions. A derivation at p is a linear map $D : C^\infty(M) \to \mathbb{R}$ such that

$$D(fg)(p) = f(p)D(g) + g(p)D(f)$$

for all $f, g \in C^\infty(M)$.

Definition

Let $f : M \to N$ be a smooth mapping from n-manifold to m-manifold. The derivative of f at p is the map $df_p : T_pM \to T_{f(p)}N$ defined by

$$(df)_p\left(\frac{dc}{dt}(0)\right) = \frac{d(f \circ c)}{dt}(0)$$

where $c : I \to M$ is a smooth curve in M with $c(0) = p$.

- Despite the derivative df is defined in terms of a curve $c(t)$, the derivative mappings is independent of the choice of $c(t)$, but only depends on $c'(0)$.
- The mapping df is a linear map from T_pM to $T_{f(p)}N$.

Regular and critical points and values

Critical points and values are used in optimization problems. We study them in general manifold settings.

Definition

Let f be a smooth map from the n-manifold M to the m-manifold N.

- A point p in M is called a critical point of f if the derivative $df_p : T_pM \to T_{f(p)}N$ is not surjective. That is, the rank of the Jacobi matrix (Jf_p) is less than the dimension n of N. The image $f(p)$ of a critical point is called a critical value of f.
- A point p in M is called a regular point of f if it is not critical. A point q in N is called a regular value of f if its inverse image $f^{-1}(q)$ contains no critical points.

Definition (Inverse image of regular value)

The inverse image $f^{-1}(p)$ of a regular value $q \in N$ of a smooth map $f : M \to N$ is a submanifold of dimension $(m - n)$ of M, unless it is empty.

- $f(x, y) = x^2 + y^2$,
- $f(x, y) = x^2 - y^2$.

Immersions and Embeddings

- Among closed surfaces the torus $T^2 = S^1 \times S^1$ can be thought of as sitting in three -dimensional space $\mathbb{R}^3$, but the Klein bottle cannot be realized there.
- This observation naturally leads us to the question: can a general n-manifold be smoothly embedded in Euclidean space $\mathbb{R}^q$?
- It is possible to embed the circle S^1 in three-dimensional space, but there is more than one way to do so. This can be generalized to the problem: are two given embeddings $f, g : M^n \to \mathbb{R}^q$ isotopic?
- In particular, the problem of classifying embeddings of circle in $\mathbb{R}^3$ or S^3 through isotopies forms a field in topology called the theory of knots.
- The problem of classifying immersions by regular homotopies is slightly easier than that of classifying embeddings by isotopies.
- Here is an example. In $\mathbb{R}^3$, is it possible to turn the sphere S^2 inside out smoothly allowing self-intersections? It seems unlikely, but a classification theorem for immersions shows that it can be done.

Let $f : M \to N$ be a C^1 map. We call f immersive at $x \in M$ if the linear map $T_x f$ is injective, and submersive if $T_x f$ is surjective. If f is immersive at every point of M it is an immersion; if it is submersive at every point, f is a submersion. f is called an embedding if f is an immersion which f maps M homeomorphically onto its image.

Theorem

Let N be a C^r manifold. A subset $A \subset N$ is a C^r submanifold if and only if A is the image of a C^r embedding.

Manifold with boundary

Definition

A smooth n-manifold with boundary is a set M with a collection of maps $\phi_\alpha : U_\alpha \to M$, with U_α a relatively open subset of the n-half space

$$H^n = \{(x_1, ..., x_n) : x_n \geq 0\},$$

satisfying the condition in the definition of a smooth manifold.

Theorem

The boundary ∂M of an n-manifold with boundary M is an $(n-1)$-manifold without boundary.

Questions?