Lecture 10: Difference and Differential Equations

Historic Notes

The study of dynamical system dates back to Newtoninan mechanics despite the word was not used as a clear branch of mathematics. Today Henri Poincaré is regard as the the founder of dynamical systems in modern sense. The essence of methodology of "dynamical system", at least in a classic literature, is to study the geometry of curves defined by differential equations that cannot be solved explicitly. The following statement is presented in "Mémoire sur les courbes définies par une équation différentielle" by Poincaré published in 1881:

"Une théorie complète des fonctions définies par les équations différentielles serait d'une grande utilité dans un grand nombre de ques- tions de Mathématiques pures ou de Mécanique. Malheureusement, il est évident que dans la grande généralité des cas qui se présentent on ne peut intégrer ces équations à l'aide des fonctions déjà connues, par exemple à l'aide des fonctions définies par les quadratures. Si l'on voulait donc se restreindre aux cas que l'on peut étudier avec des intégrales définies ou indéfinies, le champ de nos recherches serait singu lièrement diminué, et l'immense majorité des questions qui se présentent dans les applications demeureraient insolubles."

Poincaré clearly pointed out that it is necessary to study functions (mappings) defined by differential equations without trying to reduce them to simpler function (integrals). But

what are the tools to understand geometric properties of differential equations, if one does not attempt to solve them?

In this chapter, we will see **"eigenvalues, diagonalization, change of basis (coordinates)"** appearing as the most fundamental and powerful tools in understanding patterns of curves defined by differential equations.

Difference Equations

Example 1.0.1 (Cat/mouse population problem). *Suppose the cat population at month k is c_k and the mounse population at month k is m_k, and let $\mathbf{x}_k = \begin{bmatrix} c_k \\ m_k \end{bmatrix}$ denote the population vector at month K. Suppose*

$$\mathbf{x}_{k+1} = A\mathbf{x}_k$$

where

$$A = \begin{bmatrix} 0.7 & 0.2 \\ -0.6 & 1.4 \end{bmatrix}$$

and an initial population vector $\mathbf{x}_0$ is given. Then the population vector $\mathbf{x}_k$ can be computed from

$$\mathbf{x}_k = A^k \mathbf{x}_0,$$

so we want to compute A^k by diagonalizing the matrix A.

Since the charactieristic polynomial of A is

$$p(t) = t^2 - 2.1t + 1.1 = (t-1)(t-1.1),$$

the eigenvalues are 1 and 1.1. The corresponding eigenvectors are

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \text{and } \mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

and so we have the change-of-basis formula matrix

$$P = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}.$$

Then we have

$$A = P\Lambda P^{-1}$$

where

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1.1 \end{bmatrix},$$

and so

$$A^k = P\Lambda^k P^{-1} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1.1^k \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Note that if $\mathbf{x}_0 = \begin{bmatrix} c_0 \\ m_0 \end{bmatrix}$ *is the original population vector, we have*

$$\mathbf{x}_k = (2c_0 - m_0) \begin{bmatrix} 2 \\ 3 \end{bmatrix} + (-3c_0 + 2m_0)(1.1)^k \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

We can conclude, if $3c_0 = m_0$, *the second terms drops out and the population vector stays constant. If* $3c_0 < 2m_0$, *the first term is still constant, and the second term increases exponentially. If* $3c_0 > 2m_0$, *we can see that the population vector decreases exponentially, the mouse being the first to disappear.*

For general diagonalizable matrix A, the column vectors of P are the eigenvectors $\mathbf{v}_1, ..., \mathbf{v}_n$, let

$$P^{-1}\mathbf{x}_0 = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

we have

$$A^k\mathbf{x}_0 = P\Lambda^k(P^{-1})\mathbf{x}_0 = c_1\lambda_1^k\mathbf{v}_1 + ... + c_n\lambda_n^k\mathbf{v}_n.$$

Systems of Differential Equations

Another powerful application of linear algebra comes from the study of systems of ordinary differential equations (ODEs). Given an $n \times n$ matrix A and a vector $\mathbf{x}_0$, we want to find differentiable vector-valued function $\mathbf{x}(t)$ so that

$$\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0.$$

Example 1.0.2. *Suppose* $n = 1$, *the system becomes*

$$x'(t) = ax(t), \quad x(0) = x_0.$$

It is not hard to find a solution $x(t) = x_0e^{at}$, *but do we know there can be no more? Suppose* $y(t)$ *were any solution of the original problem. Then the function* $z(t) = y(t)e^{-at}$ *must satisfy the equation*

$$z'(t) = y'(t)e^{-at} + y(t)(-ae^{-at}) = 0,$$

and so $z(t)$ *must be a constant function. Since* $z(0) = y(0) = x_0$, *we have that* $y(t) = x_0e^{at}$.

Example 1.0.3. *Consider the* 2×2 *example:*

$$\frac{dx_1}{dt} = ax_1(t)$$
$$\frac{dx_2}{dt} = bx_2(t)$$

with the initial $x_1(0)$ *and* $x_2(0)$. *Since* $x_1(t)$ *and* $x_2(t)$ *appear completely independently in these equations, the solution of the system will be*

$$x_1(t) = x_1(0)e^{at}, \quad x_2(t) = x_2(0)e^{bt}.$$

In vector notation, we have

$$\mathbf{x}(t) = \begin{bmatrix} x_1(0)e^{at} \\ x_2(0)e^{bt} \end{bmatrix}$$

Recall that for any real number x, the Taylor series expansion

$$e^x = 1 + x + \frac{1}{2}x^2 + ...$$

Now, given an $n \times n$ matrix A, we define a new matrix e^A, called the exponential of A, by

$$e^A := I + A + \frac{1}{2}A^2 + ...$$

The series converges. in general, trying to evaluate this series is extremely difficult since the coefficients of A^k are not easily in terms of the coefficients of A. However, when A is a diagonalizable matrix, it is easy to compute e^A, $A^k = P\Lambda^k P^{-1}$. It holds that

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \frac{P\Lambda^k P^{-1}}{k!} = P\left(\sum_{k=0}^{\infty} \frac{\Lambda^k}{k!}\right)P^{-1} = Pe^{\Lambda}P^{-1}.$$

An immediate application is following.

Example 1.0.4. *Let* $A = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$. *Then*

$$\Lambda = \begin{bmatrix} 2 & \\ & -1 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Thus we have

$$e^{t\Lambda} = \begin{bmatrix} e^{2t} & \\ & e^{-t} \end{bmatrix} \text{ and } e^{tA} = \begin{bmatrix} e^{2t} & 0 \\ e^{2t} - e^{-t} & e^{-t} \end{bmatrix}.$$

Proposition 1.0.5. *Let A be a diagonalizable $n \times n$ matrix. The general solution of initial value problem*

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$$

is given by $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$

Even when A is not diagonalizable, we may differentiate the exponential series term-by-term to obtain

$$\frac{d}{dt}e^{tA} = Ae^{tA}.$$

Thus we have

Theorem 1.0.6. *Suppose A is an $n \times n$ matrix. Then the unique solution of the initial value problem*

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$$

is $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$.

Planar Linear Systems

We restrict our attention to the most important class of planar systems of differential equations, namely, linear systems. In the autonomous case, these systems assume the simple form

$$\begin{aligned} x_1' &= ax_1 + bx_2 \\ x_2' &= cx_1 + dx_2 \end{aligned}$$

where a, b, c, d are constants. We may abbreviate the system by using matrix A

$$\mathbf{x}' = A\mathbf{x}.$$

Note that the origin is always an equilibrium point for a linear system. To find other equilibria, we must solve the linear system of algebraic equations

$$\begin{aligned} ax_1 + bx_2 &= 0 \\ cx_1 + dx_2 &= 0. \end{aligned}$$

This system has a nonzero solution if and only if $\det A = 0$.

We are more interested in finding nonequilibrium solutions of the linear system $\mathbf{x}' = A\mathbf{x}$. The key observation is the following: suppose $\mathbf{v}_0$ for which we have

$$A\mathbf{v}_0 = \lambda\mathbf{v}_0 \text{ where } \lambda \in \mathbb{R}.$$

Then the function

$$\mathbf{x}(t) = e^{\lambda t}\mathbf{v}_0$$

is a solution of the system. To verify this, we compute

$$\mathbf{x}'(t) = \lambda e^{\lambda t}\mathbf{v}_0) = e^{\lambda t}(\lambda\mathbf{v}_0) = e^{\lambda t}(A\mathbf{v}_0) = A(e^{\lambda t}\mathbf{v}_0) = A\mathbf{x}(t).$$

The following theorem indicates an important relationship between eigenvalues, eigenvectors, and solutions of systems of differential equations.

Theorem 1.0.7. *Suppose that $\mathbf{v}_0$ is an eigenvector for the matrix A with associated eigenvalue λ. Then the function $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}_0$ is a solution of the system $\mathbf{x}' = A\mathbf{x}$.*

Example 1.0.8. *Consider*

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}$$

Then A has an eigenvector $\mathbf{v}_0 = (3,1)^\top$ with associated eigenvalue $\lambda = 2$ and $\mathbf{v}_1 = (1,-1)^\top$ is an eigenvector with associated eigenvalue $\lambda = -2$. Thus, for the system $\mathbf{x}' = A\mathbf{x}$ we know three solutions: the equilibrium solution at the origin together with

$$\mathbf{x}_1(t) = e^{2t}\begin{bmatrix} 3 \\ 1 \end{bmatrix} \text{ and } \mathbf{x}_2(t) = e^{-2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Solving Linear Systems

As we saw in the previous section, if we find two real roots λ_1 and λ_2 (with $\lambda_1 \neq \lambda_2$) fo the characteristic equation, then we may generate a pair of solutions of the system of differential equations of the form

$$\mathbf{x}_i(t) = e^{\lambda_i t}\mathbf{v}_i$$

where $\mathbf{v}_i$ is the eigenvector associated to λ_i. Note that each of these solutions if a straight-line solution. And note that, if $\lambda_i > 0$, then

$$\lim_{t\to\infty} |\mathbf{x}_i(t)| = \infty \text{ and } \lim_{t\to\infty} \mathbf{x}_i(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The magnitude of the solution $\mathbf{x}_i(t)$ increases monotonically to ∞ along the ray through $\mathbf{v}_i$ as t increases, and $\mathbf{x}_i(t)$ tends to the origin along this ray in backward time. The exact opposite situation occurs if $\lambda_i < 0$, whereas, if $\lambda_i = 0$, the solution $\mathbf{x}_i(t)$ is the constant solution $\mathbf{x}_i(t) = \mathbf{v}_i$ for all t.

We now try to find all solutions of the system given this pair of special solutions. Suppose we have two distinct real eigenvalues λ_1 and λ_2 with eigenvectors $\mathbf{v}_1, \mathbf{v}_2$. Then $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. Thus they form a basis of $\mathbb{R}^2$. So given any point $\mathbf{z}_0 \in \mathbb{R}^2$, we can find a unique pair of real numbers α and β such that

$$\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 = \mathbf{z}_0.$$

Consider the function

$$\mathbf{z}(t) = \alpha\mathbf{x}_1(t) + \beta\mathbf{x}_2(t)$$

where $\mathbf{x}_i(t)$ are the straight-line solutions. We claim that $\mathbf{z}(t)$ is a solution of $\mathbf{x}' = A\mathbf{x}$. To see this we compute

$$\mathbf{z}'(t) = \alpha\mathbf{x}_1'(t) + \beta\mathbf{x}_2'(t) = \alpha A\mathbf{x}_1(t) + \beta A\mathbf{x}_2(t) = A(\alpha\mathbf{x}_1(t) + \beta\mathbf{x}_2(t)) = A\mathbf{z}(t)$$

Moreover, $\mathbf{z}(t)$ is a solution that satisfies $\mathbf{z}(0) = \mathbf{z}_0$. Finally, we claim that $\mathbf{z}(t)$ is the unique solution that satisfies $\mathbf{z}(0) = \mathbf{z}_0$. In order to show this, we suppose that $\mathbf{y}(t)$ is another such solution with $\mathbf{y}(0) = \mathbf{z}_0$. Then we may write

$$\mathbf{y}(t) = \xi(t)\mathbf{v}_1 + \mu(t)\mathbf{v}_2$$

with $\xi(0) = \alpha$, $\mu(0) = \beta$. Hence

$$A\mathbf{y}(t) = \mathbf{y}'(t) = \xi'(t)\mathbf{v}_1 + \mu'(t)\mathbf{v}_2.$$

But

$$A\mathbf{y}(t) = \xi(t)A\mathbf{v}_1 + \mu(t)A\mathbf{v}_2 = \lambda_1\xi(t)\mathbf{v}_1 + \lambda_2\mu(t)\mathbf{v}_2.$$

Therefore, we have

$$\xi'(t) = \lambda_1\xi(t) \text{ and } \mu'(t) = \lambda_2\mu(t).$$

It follows that

$$\xi(t) = \alpha e^{\lambda_1 t}, \quad \mu(t) = \beta e^{\lambda_2 t}$$

so that $\mathbf{y}(t)$ is equal to $\mathbf{z}(t)$. We therefore have shown the following

Theorem 1.0.9. *Suppose A has a pair of real eigenvalues $\lambda_1 \neq \lambda_2$ and associated eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Then the general solution of the linear system $\mathbf{x}' = A\mathbf{x}$ is given by*

$$\mathbf{x}(t) = \alpha e^{\lambda_1 t}\mathbf{v}_1 + \beta e^{\lambda_2 t}\mathbf{v}_2.$$

Real Distinct Eigenvalues

Given the linearity principle, we may compute the general solution of any planar system. Consider $\mathbf{x}' = A\mathbf{x}$ and suppose that A has two real eigenvalues $\lambda_1 < \lambda_2$. Assuming that $\lambda_i \neq 0$, there are three cases to consider:

1. $\lambda_1 < 0 < \lambda_2$;
2. $\lambda_1 < \lambda_2 < 0$;
3. $0 < \lambda_1 < \lambda_2$.

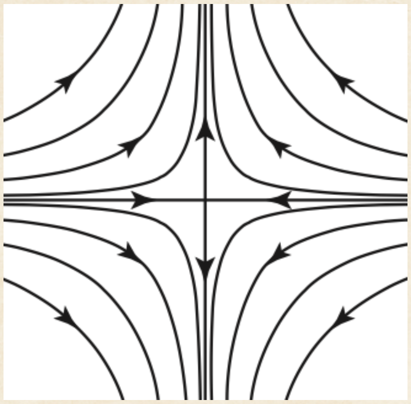
Example 1.0.10 (Saddle). *Consider the simple system $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ where*

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

with $\lambda_1 < 0 < \lambda_2$. This can be solved immediately since the system decouples into two unrelated first-order equations. The general solution has the following form:

$$\mathbf{x}(t) = ae^{\lambda_1 t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + be^{\lambda_2 t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

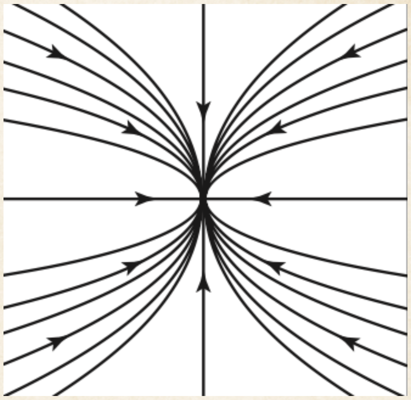
- *Since $\lambda_1 < 0$, the straightline solution of the form $\alpha e^{\lambda_1 t}(1,0)^\top$ lie on the x-axis and tend to $(0,0)^\top$ as $t \to \infty$. This axis is called the stable line.*
- *Since $\lambda_2 > 0$, the solution $\beta e^{\lambda_2 t}(0,1)^\top$ lie on the y-axis and tend away from $(0,0)^\top$ as $t \to \infty$. This axis is called the unstable line.*
- *All other solutions tend to ∞ in the direction of the unstable line as $t \to \infty$. In backward time, these solutions tend to ∞ in the direction of the stable line.*



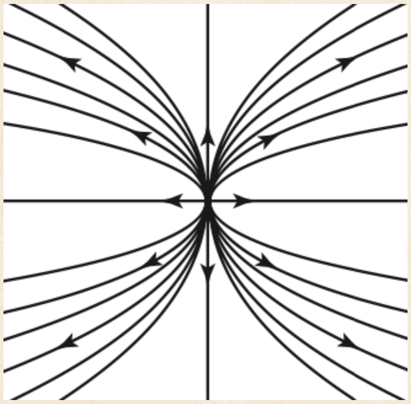
Example 1.0.11 (Sink). *Consider the case $\frac{d\mathbf{x}}{dt} = A\mathbf{x}$ when the matrix A is again diagonal, but has eigenvalues $\lambda_1 < \lambda_2 < 0$. To understand the way in which the solution curve approaches the origin, we compute the slope $\frac{dx_2}{dx_2}$ with $\beta \neq 0$.*

$$\frac{dx_2}{dx_1} = \frac{\lambda_2\beta e^{\lambda_2 t}}{\lambda_2\alpha e^{\lambda_1 t}} = \frac{\lambda_2\beta}{\lambda_1\alpha}e^{(\lambda_2-\lambda_1)t}.$$

Since $\lambda_2 - \lambda_1 > 0$, it follows that these slopes approaches $+\infty$ and $-\infty$. Thus these solutions tend to the origin tangentialy to the y-axis.



Example 1.0.12 (Source)**.** *When the matrix satisfies $0 < \lambda_2 < \lambda_1$, our vector field may be regarded as the negative of the previous example. The general solution and the phase portrait remain the same, except that all solutions now tend away from $(0, 0)^\top$ along the same path.*



Complex Eigenvalues

It may happen that the roots of the characteristic polynomial are complex numbers. When the matrix A has complex eigenvalues, we no longer have straight line solutions. However, we can still derive the general solution.

Example 1.0.13 (Center)**.** *Consider $\mathbf{x}' = A\mathbf{x}$ with*

$$A = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}$$

and $\beta \neq 0$. The characteristic polynomial is $\lambda^2 + \beta^2 = 0$, so the eigenvalues are now the impaginary numbers $\pm i\beta$. Without worrying about the "complex vectors", we try to find the eigenvector corresponding to $\lambda = i\beta$. We therefore solve

$$\begin{bmatrix} -i\beta & \beta \\ -\beta & -i\beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

since the second equation is redundant. Thus we find a complex eigenvector $(1, i)^\top$, and so the function

$$\mathbf{x}(t) = e^{i\beta t} \begin{bmatrix} 1 \\ i \end{bmatrix}$$

is a complex solution.

With the help of Euler's formula,

$$e^{i\beta t} = \cos\beta t + i \sin\beta t$$

we can rewrite the solution as

$$\mathbf{x}(t) = \begin{bmatrix} \cos\beta t + i\sin\beta t \\ i(\cos\beta t + i\sin\beta t) \end{bmatrix} = \begin{bmatrix} \cos\beta t + i\sin\beta t \\ -\sin\beta t + i\cos\beta t \end{bmatrix}.$$

By breaking $\mathbf{x}(t)$ *into its real and imaginary parts, we have*

$$\mathbf{x}(t) = \mathbf{x}_{Re}(t) + i\mathbf{x}_{Im}(t)$$

where

$$\mathbf{x}_{Re}(t) = \begin{bmatrix} \cos \beta t \\ -\sin \beta t \end{bmatrix}, \quad \mathbf{x}_{Im}(t) = \begin{bmatrix} \sin \beta t \\ \cos \beta t \end{bmatrix}.$$

But now we can verify that both $\mathbf{x}_{\mathrm{Re}}(t)$ *and* $\mathbf{x}_{\mathrm{Im}}(t)$ *are solutions of the original system.*

$$\mathbf{x}'_{\mathrm{Re}}(t) + i\mathbf{x}'_{\mathrm{Im}}(t) = \mathbf{x}'(t) = A\mathbf{x}(t) = A(\mathbf{x}_{Real}(t) + i\mathbf{x}_{Im}(t)) = A\mathbf{x}_{\mathrm{Re}}(t) + iA\mathbf{x}_{Imaginary}(t).$$

Equating the real and imaginary parts of this equation yields

$$\mathbf{x}'_{\mathrm{Re}} = A\mathbf{x}_{\mathrm{Re}} \text{ and } \mathbf{x}'_{\mathrm{Im}} = A\mathbf{x}_{\mathrm{Im}}$$

which shows that both are indeed solutions. Moreover, since

$$\mathbf{x}_{\mathrm{Re}}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \mathbf{x}_{\mathrm{Im}}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The linear combination of them is

$$\mathbf{x}(t) = c_1\mathbf{x}_{\Re}(t) + c_2\mathbf{x}_{\Im}(t)$$

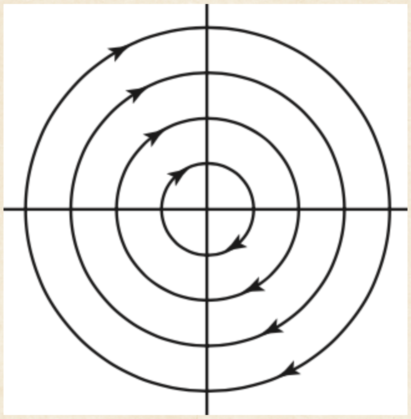
We next show that this is the general solution of this equation. Suppose that this is not the only solution, let

$$\mathbf{y}(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}$$

be another solution. Consider the complex function

$$f(t) = (u(t) + iv(t))e^{i\beta t}.$$

Differentiating this expression and using the fact that $\mathbf{y}(t)$ *is a solution of the equation yields* $f'(t) = 0$*. Hence* $u(t) + iv(t)$ *is a complex constant times* $e^{-i\beta t}$*. It follows that* $\mathbf{y}(t)$ *is a linear combination of* $\mathbf{x}_{\Re}(t)$ *and* $\mathbf{x}_{\Im}(t)$*. Furthermore, note that each of these solutions is a periodic function with period* $\frac{2\pi}{\beta}$*. Indeed, the phase portrait shows that all solutions lie on circles centered at the origin. This type of system is called a* center.



Example 1.0.14 (Spiral sink, spiral source)**.** *More generally, consider* $\mathbf{x}' = A\mathbf{x}$ *for*

$$A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

and $\alpha, \beta \neq 0$*. The characteristic equation is*

$$\lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2,$$

so the eigenvalues are

$$\lambda_1 = \alpha + i\beta, \quad \lambda_2 = \alpha - i\beta$$

Gradient Systems

A gradient system on $\mathbb{R}^n$ is a system of differential equations of the form

$$\frac{d\mathbf{x}}{dt} = -\nabla f(\mathbf{x})$$

where $f : \mathbb{R}^n \to \mathbb{R}$ is a C^∞ function. The vector field ∇f is called the gradient of f. Gradient system have special properties that make their phase portraits rather simple. The following equality is fundamental:

$$Df_{\mathbf{x}}(\mathbf{y}) = \nabla f(\mathbf{x}) \cdot \mathbf{y}.$$

This says that the derivative of f at $\mathbf{x}$ evaluated at $\mathbf{y}$ is given by the dot product of the vectors $\nabla f(\mathbf{x})$ and $\mathbf{y}$. This follows from the formula

$$Df_{\mathbf{x}}(\mathbf{y}) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} y_j.$$

Let $\mathbf{x}(t)$ be a solution of the gradient system with $\mathbf{x}(0) = \mathbf{x}_0$, and let $\dot{f} : \mathbb{R}^n \to \mathbb{R}$ be the derivative of f along this solution. That is,

$$\dot{f}(\mathbf{x}) = \frac{d}{dt} f(\mathbf{x}(t)).$$

Proposition 1.0.15. *The function f is a Lyapunov function for the system $\mathbf{x}' = -\nabla f(\mathbf{x})$. Moreover, $\dot{f}(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{x}$ is an equilibrium point.*

Proof. By the chain rule we have

$$\dot{f}(\mathbf{x}) = Df_{\mathbf{x}}(\mathbf{x}') = \nabla f(\mathbf{x}) \cdot (-\nabla f(\mathbf{x})) = -\left\| \nabla f(\mathbf{x}) \right\|^2 \leq 0.$$

In particular, $\dot{f}(\mathbf{x}) = \mathbf{0}$ if and only if $\nabla f(\mathbf{x}) = \mathbf{0}$. □

An immediate consequence of this is the fact that if $\mathbf{x}^*$ is an isolated minimum of f, then $\mathbf{x}^*$ is an asymptotically stable equilibrium of the gradient system.

Homework

Solve the following systems with arbitrary initial condition $\mathbf{x}(0) = (x_1(0), x_2(0))$.

1. $$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \mathbf{x}$$

2. $$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \mathbf{x}$$