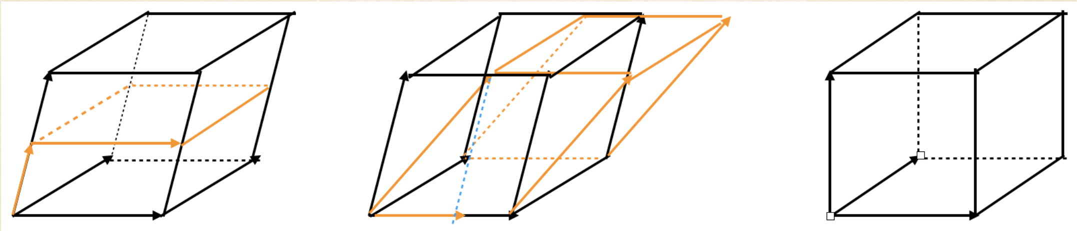
Lecture 6: Determinant

Proposition 1.0.1. *Let A be an $n \times n$ matrix.*

1. *Let A' be obtained from A by exchanging two rows. Then $\det A' = -\det A$.*
2. *Let A' be obtained from A by multiplying some row by the scalar c. Then $\det A' = c \det A$.*
3. *Let A' be obtained from A by adding a multiple of one row to another. Then $\det A' = \det A$.*
4. *Last, $\det I_n = 1$.*

The following figure illustrates the property of 2,3 and 4.



Theorem 1.0.2. *Let A be a square matrix. Then A is nonsingular iff $\det A \neq 0$.*

Proof. Suppose A is nonsingular. Then its reduced echelon form is I_n, which is obtained from a sequence of row operations. If we keep track of the effect on the determinant, it is a multiplication of $\det I$ and nonzero numbers. So $\det A \neq 0$.

Conversely, suppose A is singular. Then its echelon form U has a row of zeroes, and then $\det U = 0$. It follows that $\det A = 0$. □

The following result is useful in computational and theoretical grounds.

Proposition 1.0.3. *If A is an upper (lower) triangular $n \times n$ matrix, then $\det A = a_{11}a_{22}...a_{nn}$.*

Proof. If $a_{ii} = 0$ for some i, then A is singular and so $\det A = 0$, and the desired equality holds. Now assume all the a_{ii} are nonzero. Let $\mathbf{A}_i$ be the ith row vector of A, as usual, and write $\mathbf{A}_i = a_{ii}\mathbf{B}_i$, where the ith entry of $\mathbf{B}_i$ is 1. Then, letting B be the matrix with rows $\mathbf{B}_i$ and using property 2 of the determinant, we have $\det A = a_{11}...a_{nn} \det B$. Now B is an upper triangle matrix with 1's on the diagonal, so using property 3, we can use the pivot to clear out the upper entries without changing the determinant. Thus, $\det B = \det I = 1$. And then we have $\det A = a_{11}...a_{nn}$. □

There is also "product rule" for determinants.

Proposition 1.0.4. *Let E be an elementary matrix, and let A be an arbitrary square matrix. Then*

$$\det(EA) = \det E \det A.$$

Theorem 1.0.5. *Let A and B be $n \times n$ matrices. Then*

$$\det(AB) = \det A \det B.$$

Proof. Suppose A is singular, so that there is some nontrivial linear relation among its row vectors:

$$c_1\mathbf{A}_1 + ... + c_n\mathbf{A}_n = \mathbf{0}.$$

Then, multiplying by B on the right, we find that

$$c_1(\mathbf{A}_1 B) + ... + c_n(\mathbf{A}_n B) = \mathbf{0},$$

from which we conclude that there is the same nontrivial linear relation among the row vectors of AB, and so AB is singular as well. We can have that both $\det A = 0$ and $\det AB = 0$.

Otherwise, if A is nonsingular, we know that we can write A as a product of elementary matrices. We now apply the previous proposition twice. First, we have

$$\det A = \det(E_m ... E_1) = \det E_m ... \det E_1.$$

Then we have

$$\det AB = \det(E_m ... E_1 B) = \det E_m ... \det E_1 \det B = (\det E_m ... \det E_1) \det B = \det A \det B.$$

□

Corollary 1.0.6. *If A is nonsingular, then* $\det(A^{-1}) = \frac{1}{\det A}$.

Proof. From the equation $AA^{-1} = I$, we have that

$$\det A \det(A^{-1}) = 1$$

so

$$\det(A^{-1}) = \frac{1}{\det A}.$$

□

Recall that B is similar to A if $B = P^{-1}AP$ for some invertible matrix P. A fundamental consequence of the product rule is the fact that similar matrices have the same determinant.

$$\det(P^{-1}AP) = \det(P^{-1}) \det(AP) = \det(P^{-1}) \det A \det P = \det A.$$

As a result, we can define determinant for linear transformation of finite dimensional vector space $T : V \to V$. One writes down the matrix A for T with respect to *any* basis and defines $\det T = \det A$. The change of basis formula tells us that any two matrices representing T are similar and hence, have the same determinant. Furthermore $\det T$ has a nice geometric meaning: It gives the factor by which signed volume is distorted under the mapping by T.

Proposition 1.0.7. *Let A be a square matrix. Then*

$$\det(A^\top) = \det A.$$

The third property of determinant can be rewritten as follows.

Suppose the ith row of the matrix A is written as a sum of two vectors, $A_i = A'_i + A''_i$. Let A' denote the matrix with A'_i as its ith row and all other rows the same as those of A. Then

$$\det A = \det A' + \det A''.$$

Lemma 1.0.8. *If two rows of a matrix A are equal, then* $\det A = 0$.

Proposition 1.0.9. *Let A be an $n \times n$ matrix, and let B be the matrix obtained by adding a multiple of one row of A to another. Then* $\det B = \det A$.

Proof. Suppose B is obtained from A by replacing the i th row by its sum with c times the jth row; i.e.,

$$B_i = A_i + cA_j$$

By the linearity in rows,

$$\det B = \det A + \det A',$$

where $A'_i = cA_j$ and all the other rows of A' are the corresponding rows of A. If we define the matrix A'' by setting $A''_i = A_j$ and keeping all the other rows the same. Then property 2 guarantees that

$$\det A' = c \det A''.$$

But two rows of the matrix A'' are identical, so by previous lemma, $\det A'' = 0$. Therefore, $\det B = \det A$.

□

Cofactors and Cramer's Rule

Given the 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Use the row linearity in the first row to calculate

$$\begin{aligned} \det A &= a_{11} \det \begin{bmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + a_{12} \det \begin{bmatrix} 0 & 1 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ &= a_{11} \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} + a_{12} \det \begin{bmatrix} 0 & 1 & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & 0 & a_{23} \\ a_{31} & 0 & a_{33} \end{bmatrix} + a_{13} \det \begin{bmatrix} 0 & 0 & 1 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & 0 \end{bmatrix} \end{aligned}$$

The preceding calculations suggest a general recursive formula. Given an $n \times n$ matrix A with $n \geq 2$, denote by A_{ij} the $(n-1) \times (n-1)$ matrix obtained by deleting the ith row and the jth column from A. Define the ijth *cofactor* of the matrix to be

$$C_{ij} = (-1)^{i+j} \det A_{ij}.$$

Why $(-1)^{i+j}$**?.** To move the ith row to the top, without otherwise changing the order of the rows, requires switching pairs of rows $i-1$ times; this gives a sign of $(-1)^{i-1}$. We then alternate signs as we proceed from column to column, the jth column contributing a sign of $(-1)^{j-1}$. Thus, in the expansion, $\det A_{ij}$ appears with a factor of $(-1)^{i-1}(-1)^{j-1} = (-1)^{i+j}$.

Proposition 1.0.10. *Let A be an $n \times n$ matrix. Then for any fixed i, we have*

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij}.$$

Complexity. Despite expansion in cofactors is an important theoretical tool, it is a computational nightmare. Computation complexity of calculating an $n \times n$ determinant by expanding in cofactors requires approximately $n!$ multiplications and additions.

We conclude this section with a few classic formulas. the first is particularly useful for solving 2×2 systems of equations and maybe useful for larger n.

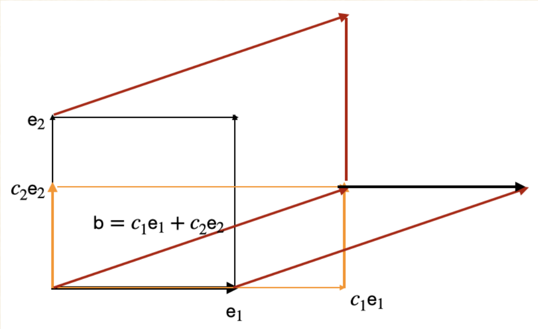
Proposition 1.0.11 (Cramer's Rule.). *Let A be a nonsingular $n \times n$ matrix, and let $\mathbf{b} \in \mathbb{R}^n$. Then the ith coordinate of the vector $\mathbf{x}$ solving $A\mathbf{x} = \mathbf{b}$ is*

$$x_i = \frac{\det B_i}{\det A},$$

where B_i is the matrix obtained by replacing the ith column of A by the vector $\mathbf{b}$.

This result's geometric interpretation is clear as well, if we can use the special case of orthonormal basis as an example. The picture indicates that the area of parallelogram spanned by $(\mathbf{e}_1, \mathbf{b})$ equals c_2 times the area of the square spanned by $(\mathbf{e}_1, \mathbf{e}_2)$, and the area of the parallelogram spanned by $(\mathbf{b}, \mathbf{e}_2)$ equals c_1 times the area of the square spanned by $(\mathbf{e}_1, \mathbf{e}_2)$. Formally, we can obtain Cramer's rule by comparing the areas of the red framed parallelograms and the unit square, i.e.,

$$c_1 = \frac{\text{Area}(\mathbf{b}, \mathbf{e}_2)}{\text{Area}(\mathbf{e}_1, \mathbf{e}_2)}, \quad c_2 = \frac{\text{Area}(\mathbf{e}_1, \mathbf{b})}{\text{Area}(\mathbf{e}_1, \mathbf{e}_2)}$$



Proof. We calculate the determinant of the matrix obtained by replacing the ith column of A by $\mathbf{b} = A\mathbf{x} = x_1\mathbf{a}_1 + ... + x_n\mathbf{a}_n$:

$$\det B_i = \det \begin{bmatrix} | & & | & & | \\ \mathbf{a}_1 & \dots & x_1\mathbf{a}_1 + ... + x_n\mathbf{a}_n & \dots & \mathbf{a}_n \\ | & & | & & | \end{bmatrix} = \det \begin{bmatrix} | & & | & & | \\ \mathbf{a}_1 & \dots & x_i\mathbf{a}_i & \dots & \mathbf{a}_n \\ | & & | & & | \end{bmatrix} = x_i \det A.$$

□

Proposition 1.0.12. *Let A be a nonsingular matrix, and let $C = [C_{ij}]$ be the matrix of its cofactors. Then*

$$A^{-1} = \frac{1}{\det A} C^{\top}.$$

Homework

1. Let A be an $n \times n$ matrix. Show that

$$\det \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{bmatrix} = \det A$$

2. Show that

$$\det \begin{bmatrix} 1 & 1 & 1 \\ b & c & d \\ b^2 & c^2 & d^2 \end{bmatrix} = (c-b)(d-b)(d-c)$$

3. Evaluate

$$\det \begin{bmatrix} 1 & t_1 & t_1^2 & \dots & t_1^k \\ 1 & t_2 & t_2^2 & \dots & t_2^k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & t_{k+1} & t_{k+1}^2 & \dots & t_{k+1}^k \end{bmatrix}$$

4. Suppose $A \in M_{k\times k}, B \in M_{k\times l}$, and $D \in M_{l\times l}$. Prove that

$$\det \begin{bmatrix} A & B \\ O & D \end{bmatrix} = \det A \det D$$

5. Suppose $C \in M_{l\times k}$. Prove that if A is invertible, then

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det A \det(D - CA^{-1}B).$$

6. If we assume that $k = l$ and $AC = CA$, then

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - CB).$$