Lecture 5: Linear Transformations and Change of Basis

Definition 1.0.1. *Let V be a finite-dimensional vector space and let $T : V \to V$ be a linear transformation. Let $\mathcal{B} = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ be an ordered basis for V. Define numbers a_{ij}, $i = 1, ..., n$, $j = 1, ..., n$, by*

$$T(\mathbf{v}_j) = a_{1j}\mathbf{v}_1 + a_{2j}\mathbf{v}_2 + ... + a_{nj}\mathbf{v}_n.$$

Then we define $A = [a_{ij}]$ to be the matrix for T with respect to $\mathcal{B}$*, also denote $[T]_{\mathcal{B}}$. As before we have*

$$A = \begin{bmatrix} | & | & \cdots & | \\ T(\mathbf{v}_1) & T(\mathbf{v}_2) & \cdots & T(\mathbf{v}_n) \\ | & | & \cdots & | \end{bmatrix},$$

where the column vectors are the coordinates of the vectors with respect to the basis $\mathcal{B}$.

Given a finite-dimensional vector space V and an ordered basis $\mathcal{B}\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ for V, we can define a linear transformation

$$C_{\mathcal{B}} : V \to \mathbb{R}^n,$$

which assigns to each vector $\mathbf{v}$ its vector of coordinates with respect to the basis $\mathcal{B}$. That is

$$C_{\mathcal{B}}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n) = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

When $\mathcal{B}$ is the standard basis $\mathcal{E}$ for $\mathbb{R}^n$, this is

$$C_{\mathcal{E}}(\mathbf{x}) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

With this notation, we can write $[T]_{\mathcal{B}}$ in terms of the mapping $C_{\mathcal{B}}$.

$$[T]_{\mathcal{B}} = \begin{bmatrix} | & | & & | \\ C_{\mathcal{B}}(T(\mathbf{v}_1)) & C_{\mathcal{B}}(T(\mathbf{v}_2)) & \cdots & C_{\mathcal{B}}(T(\mathbf{v}_n)) \\ | & | & & | \end{bmatrix}$$

Throughout this notes, we will use $\mathcal{E} = \{\mathbf{e}_1, ..., \mathbf{e}_n\}$ for standard basis.

Suppose that we have a linear transformation $T : V \to V$ and two ordered bases $\mathcal{B} = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ and $\mathcal{B}' = \{\mathbf{v}'_1, ..., \mathbf{v}'_n\}$ for V. Let $A_{\text{old}} = [T]_{\mathcal{B}}$ be the matrix for T with respect to the “old " basis $\mathcal{B}$, and let $A_{\text{new}} = [T]_{\mathcal{B}'}$ be the matrix for T with respect to the “new" basis $\mathcal{B}'$. The fundamental question is to compute A_{new} if we know A_{old}. Define the *change of basis matrix* P to be the matrix whose column vectors are the coordinates of the new basis vectors with respect to the old basis

$$\mathbf{v}'_j = p_{1j}\mathbf{v}_1 + p_{2j}\mathbf{v}_2 + ... + p_{nj}\mathbf{v}_n.$$

When $\mathcal{B}$ is the standard basis, we have

$$P = \begin{bmatrix} | & | & & | \\ \mathbf{v}'_1 & \mathbf{v}'_2 & \cdots & \mathbf{v}'_n \\ | & | & & | \end{bmatrix}$$

Theorem 1.0.2 (Change of basis formula, standard basis). *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix $[T]_{\mathcal{E}}$. Let $\mathcal{B} = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ be basis for $\mathbb{R}^n$ and let $[T]_{\mathcal{B}}$ be the matrix for T with respect to $\mathcal{B}$. Let P be the $n \times n$ matrix whose columns are given by the vectors $\mathbf{v}_1, ..., \mathbf{v}_n$. Then we have*

$$[T]_{\mathcal{E}}P = P[T]_{\mathcal{B}}.$$

Proof. The jth column of P is he vector $\mathbf{v}_j$, i.e., its coordinate vector $C_{\mathcal{E}}(\mathbf{v}_j)$ with respect to the standard basis. Therefore, the jth column vector of the matrix product $[T]_{\mathcal{E}}P$ is the standard coordinate vector of $T(\mathbf{v}_j)$. On the other hand, the jth column of $[T]_{\mathcal{B}}$ is the coordinate vector, $C_{\mathcal{B}}(T(\mathbf{v}_j))$, of $T(\mathbf{v}_j)$ with respect to the basis $\mathcal{B}$. If the jth column of $[T]_{\mathcal{B}}$ is $(a_1, ..., a_n)^\top$, then $T(\mathbf{v}_j) = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_n\mathbf{v}_n$. But we also know that

$$P\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + ... + a_n\mathbf{v}_n.$$

That is, the jth column of $P[T]_{\mathcal{B}}$ is exactly the linear combination of the columns of P needed to give the standard coordinate vector of $T(\mathbf{v}_j)$. □

Theorem 1.0.3 (Change of basis formula, general basis). *Let $T : V \to V$ be a linear transformation, and let $\mathcal{B} = \{\mathbf{v}_1, ..., \mathbf{v}_n\}$ and $\mathcal{B}' = \{\mathbf{v}'_1, ..., \mathbf{v}'_n\}$ be ordered bases for V. If $[T]_{\mathcal{B}}$ and $[T]_{\mathcal{B}'}$ are the matrices for T with respect to the respective bases and P is the change of basis matrix (whose columns are the coordinates of the new basis vectors with respect to the old basis), then we have*

$$[T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}}P.$$

Definition 1.0.4. *Two matrices A and B are called* similar *if $B = P^{-1}AP$ for some invertible matrix.*

Theorem tells us that any two matrices representing the same linear map $T : V \to V$ are similar.

Proof. Given a vector $\mathbf{v} \in V$, denote by $\mathbf{x}$ and $\mathbf{x}'$, respectively, its coordinate vectors with respect to the bases $\mathcal{B}$ and $\mathcal{B}'$. We need to prove the relation between $\mathbf{x}$ and $\mathbf{x}'$:

$$\mathbf{x} = P\mathbf{x}'$$

Using the equations

$$\mathbf{v} = \sum_{i=1}^{n} x_i\mathbf{v}_i$$

and

$$\mathbf{v} = \sum_{j=1}^{n} x'_j\mathbf{v}'_j = \sum_{j=1}^{n} x'_j\left(\sum_{i=1}^{n} p_{ij}\mathbf{v}_i\right) = \sum_{i=1}^{n}\left(\sum_{j=1}^{n} p_{ij}x'_j\right)\mathbf{v}_i,$$

so we have

$$x_i = \sum_{j=1}^{n} p_{ij}x'_j.$$

If $T(\mathbf{v}) = \mathbf{w}$, let $\mathbf{y}$ and $\mathbf{y}'$ denote the coordinate vectors of $\mathbf{w}$ with respect to bases $\mathcal{B}$ and $\mathcal{B}'$. Now compare the equations

$$\mathbf{y}' = [T]_{\mathcal{B}'}\mathbf{x}' \text{ and } \mathbf{y} = [T]_{\mathcal{B}}\mathbf{x},$$

using

$$\mathbf{y} = \mathbf{y}' \text{ and } \mathbf{x} = P\mathbf{x}'$$

On one hand, we have

$$\mathbf{y} = P\mathbf{y}' = P([T]_{\mathcal{B}'}\mathbf{x}') = (P[T]_{\mathcal{B}'}\mathbf{x}'),$$

and on the other hand,

$$\mathbf{y} = [T]_{\mathcal{B}}\mathbf{x} = [T]_{\mathcal{B}}(P\mathbf{x}') = ([T]_{\mathcal{B}}P)\mathbf{x}',$$

from which we have that

$$[T]_{\mathcal{B}}P = P[T]_{\mathcal{B}'} \text{ or equivalently } [T]_{\mathcal{B}'} = P^{-1}[T]_{\mathcal{B}}P.$$

□

Example 1.0.5. *Given the matrix*

$$A = [T] = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$$

of a linear transformation T on $\mathbb{R}^2$ with respect to the standard basis. Calculate the matrix $[T]_{\mathcal{B}'}$ with respect to the new basis $\mathcal{B}' = \{\mathbf{v}_1, \mathbf{v}_2\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

The change of basis matrix is

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} \text{ and } P^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

Then

$$[T]_{\mathcal{B}'} = P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

Orthogonality

Definition 1.0.6. *Let $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathbb{R}^n$. We say $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is an* orthogonal set *of vetors provided $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$. We say $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is an* orthogonal basis *for a subspace V if $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is both a basis for V and an orthogonal set. Moreover, we say $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is an* orthonormal basis *for V if it is an orthogonal basis consisting of unit vectors.*

The first reason that orthogonal sets of vectors are important is the following:

Proposition 1.0.7. *Let $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathbb{R}^m$. If $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is an orthogonal set of nonzero vectors, then $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is linearly indepdent.*

Proof. Suppose

$$c_1\mathbf{v}_1 + ... + c_k\mathbf{v}_k = \mathbf{0}.$$

We need to show that $c_1 = c_2 = ... = c_k = 0$. For any $i = 1, ..., k$, we take the dot product of this equation with $\mathbf{v}_i$, obtain

$$c_1(\mathbf{v}_1 \cdot \mathbf{v}_i) + ... + c_k(\mathbf{v}_k \cdot \mathbf{v}_i) = 0.$$

Since $\mathbf{v}_i$ is assumed to be orthogonal to all the other $vecv_j$'s, i.e.,

$$\mathbf{v}_1 \cdot \mathbf{v}_i = 0,$$

then we have

$$c_i \left\|\mathbf{v}_i\right\|^2 = 0.$$

Since $\mathbf{v}_i \neq \mathbf{0}$, it follows that $c_i = 0$. Repeat this argument for all $1, ..., k$, we have $c_1 = ...c_k = 0$. ☐

Lemma 1.0.8. *Suppose $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is a basis for V. Then the equation*

$$\mathbf{x} = \sum_{i=1}^{k} \text{proj}_{\mathbf{v}_i}\mathbf{x} = \sum_{i=1}^{k} \frac{\mathbf{x}_i \cdot \mathbf{v}_i}{\left\|\mathbf{v}_i\right\|^2}\mathbf{v}_i$$

holds for all $\mathbf{x} \in V$ if and only if $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is an orthogonal basis for V.

Proof. Suppose $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is an orthogonal basis for V. Then there are scalars $c_1, ..., c_k$ so that

$$\mathbf{x} = c_1\mathbf{v}_1 + ... + c_k\mathbf{v}_k.$$

We take the dot product of this equation with $\mathbf{v}_i$:

$$\mathbf{x} \cdot \mathbf{v}_i = c_1(\mathbf{v} \cdot \mathbf{v}_i) + ... + c_k(\mathbf{v}_k \cdot \mathbf{v}_i) = c_i \left\|\mathbf{v}_i\right\|^2,$$

and so

$$c_i = \frac{\mathbf{x} \cdot \mathbf{v}_i}{\left\|\mathbf{v}_i\right\|^2}.$$

Conversely, suppose that every vector $\mathbf{x}$ is the sum of its projections on $\mathbf{v}_1, ..., \mathbf{v}_k$. Consider $\mathbf{v}_1$ for example, since $\mathbf{v}_1$ is a vector in V, thus by assumption, it naturally satisfies

$$\mathbf{v}_1 = \sum_{i=1}^{k} \text{proj}_{\mathbf{v}_i} \mathbf{v}_1 = \sum_{i=1}^{k} \frac{\mathbf{v}_1 \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2} \mathbf{v}_i.$$

On the other hand, $\mathbf{v}_1, ..., \mathbf{v}_k$ form a basis of V as a default setting, so each vector has a unique expression in terms of $\mathbf{v}_i$'s. So we ahve

$$\mathbf{v}_1 \cdot \mathbf{v}_i = 0 \quad \text{for all } i = 2, ..., k.$$

A similar argument shows that $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \neq j$, and the proof completes. □

We recall that whenever $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is a basis for V, every vector $\mathbf{x} \in V$ can be written uniquely as a linear combination

$$\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k,$$

where the coefficients c_i's are called the *coordinates* of $\mathbf{x}$ with repsect to the basis $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$. When $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ forms an orthogonal basis for V, the dot product gives the coordinates of $\mathbf{x}$.

Proposition 1.0.9. *Let $V \subset \mathbb{R}^m$ be a k-dimensional subspace. The equation*

$$\text{proj}_V \mathbf{b} = \sum_{i=1}^{k} \text{proj}_{\mathbf{v}_i} \mathbf{b} = \sum_{i=1}^{k} \frac{\mathbf{b} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2} \mathbf{v}_i$$

holds for all $\mathbf{b} \in \mathbb{R}^m$ if and only if $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is an orthogonal basis for V.

Proof. Assume $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is an orthogonal basis for V and let $\mathbf{v} \in \mathbb{R}^m$. Write

$$\mathbf{b} = \mathbf{p} + (\mathbf{b} - \mathbf{p}),$$

where

$$\mathbf{p} = \text{proj}_V \mathbf{b}.$$

Then, since $\mathbf{p} \in V$, it follows that

$$\mathbf{p} = \sum_{i=1}^{k} \frac{\mathbf{p} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2} \mathbf{v}_i.$$

Moreover, for $i = 1, ..., k$, we have $\mathbf{b} \cdot \mathbf{v}_i = \mathbf{p} \cdot \mathbf{v}_i$, since $\mathbf{b} - \mathbf{p} \in V^{\perp}$. Thus

$$\text{proj}_V \mathbf{b} = \mathbf{p} = \sum_{i=1}^{k} \text{proj}_{\mathbf{v}_i} \mathbf{p} = \sum_{i=1}^{k} \frac{\mathbf{p} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2} \mathbf{v}_i = \sum_{i=1}^{k} \frac{\mathbf{b} \cdot \mathbf{v}_i}{\|\mathbf{v}_i\|^2} \mathbf{v}_i = \sum_{i=1}^{k} \text{proj}_{\mathbf{v}_i} \mathbf{b}.$$

Conversely, suppose

$$\text{proj}_V \mathbf{b} = \sum_{i=1}^{k} \text{proj}_{\mathbf{v}_i} \mathbf{b}$$

for all $\mathbf{b} \in \mathbb{R}^m$. In particular, when $\mathbf{v} \in V$, have that $\mathbf{b} = \text{proj}_V \mathbf{b}$ can be written as a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_k$, so these vectors span V. Since V is k-dimensional, $\mathbf{v}_1, ..., \mathbf{v}_k$ form a basis for V. The previous lemma shows that it must be an orthogonal basis. □

We next develop an algorithm for transforming a given basis $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ for a subspace into an orthogonal basis $\{\mathbf{w}_1, ..., \mathbf{w}_k\}$. The idea is the following. We set

$$\mathbf{w}_1 = \mathbf{v}_1.$$

If $\mathbf{v}_2$ is orthogonal to $\mathbf{w}_1$, then we set $\mathbf{w}_2 = \mathbf{v}_2$. If $\mathbf{v}_2$ is not orthogonal to $\mathbf{w}_1$, then we set

$$\mathbf{w}_= \mathbf{v}_2 - \text{proj}_{\mathbf{w}_1} \mathbf{v}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2} \mathbf{w}_1.$$

Then so far by construction, $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal and $\text{Span}(\mathbf{w}_1, \mathbf{w}_2) \subset \text{Span}(\mathbf{v}_1, \mathbf{v}_2)$. Since $\mathbf{w}_2 \neq \mathbf{0}$, $\{\mathbf{w}_1, \mathbf{w}_2\}$ must be linearly independent and therefore give a basis for $\text{Span}(\mathbf{v}_1, \mathbf{v}_2)$. We continue, replacing $\mathbf{v}_3$ by its part orthogonal to the plane spanned by $\mathbf{w}_1$ and $\mathbf{w}_2$:

$$\mathbf{w}_3 = \mathbf{v}_3 - \text{proj}_{\text{Span}(\mathbf{w}_1, \mathbf{w}_2)} \mathbf{v}_3 = \mathbf{v}_3 - \text{proj}_{\mathbf{w}_1} \mathbf{v}_3 - \text{proj}_{\mathbf{w}_2} \mathbf{v}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2} \mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\|\mathbf{w}_2\|^2} \mathbf{w}_2.$$

The same discussion can be continued as well. Thus we end up with the following theorem formalizing this process of finding orthogonal basis given an arbitary basis.

Gram-Schmidt process.

Theorem 1.0.10. *Given a basis* $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ *for an inner product space* V*, we obtain an orthogonal basis* $\{\mathbf{w}_1, ..., \mathbf{w}_k\}$ *for* V *as follows:*

$$\begin{aligned} \mathbf{w}_1 &= \mathbf{v}_1 \\ \mathbf{w}_2 &= \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 \\ &\vdots \end{aligned}$$

and, assuming $\mathbf{w}_1, ..., \mathbf{w}_j$ *have been defined,*

$$\begin{aligned} \mathbf{w}_{j+1} &= \mathbf{v}_{j+1} - \frac{\mathbf{v}_{j+1} \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_{j+1} \cdot \mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 - ... - \frac{\mathbf{v}_{j+1} \cdot \mathbf{w}_j}{\|\mathbf{w}_j\|^2}\mathbf{w}_j \\ &\vdots \\ \mathbf{w}_k &= \mathbf{v}_k - \frac{\mathbf{v}_k \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_k \cdot \mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 - ... - \frac{\mathbf{v}_k \cdot \mathbf{w}_{k-1}}{\|\mathbf{w}_{k-1}\|^2}\mathbf{w}_{k-1}. \end{aligned}$$

If we so desire, we can arrange for an orthonormal basis by dividing each of $\mathbf{w}_1, ..., \mathbf{w}_k$ by its respective length:

$$\mathbf{q}_1 = \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}, \quad \mathbf{q}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}, \quad ... \quad \mathbf{q}_k = \frac{\mathbf{w}_k}{\|\mathbf{w}_k\|}.$$

Example 1.0.11. *Let* $\mathbf{v}_1 = (1,1,1,1)^\top, \mathbf{v}_2 = (3,1,-1,1)^\top$ *and* $\mathbf{v}_3 = (1,1,3,3)$*. Use Gram-Schmit process to give an orthogonal basis for* $V = \mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \subset \mathbb{R}^4$*. We can take*

$$\begin{aligned} \mathbf{w}_1 &= \mathbf{v}_1 = (1,1,1,1); \\ \mathbf{w}_2 &= \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 = (2,0,-2,0) \\ \mathbf{w}_3 &= \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\|\mathbf{w}_1\|^2}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\|\mathbf{w}_2\|^2}\mathbf{w}_2 = (0,-1,0,1) \end{aligned}$$

We can furthermore make them a orthonormal basis by normalization with respect to norms.

Homework

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given by reflecting across the plane $x_1 - 2x_2 + 2x_3 = 0$. Use the change-of-basis formula to find its standard matrix.
2. Let $V \subset \mathbb{R}^3$ be the subspace defined by

$$V = \{(x_1, x_2, x_3)^\top : x_1 - x_2 + x_3 = 0\}.$$

   Find the standard matrix for each of the following linear transformations:

   (a) projection on V
   (b) reflection across V
   (c) rotation of V through angle $\pi/6$

3. Let

$$\mathbf{a} = \begin{bmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{bmatrix} \quad 0 \le \phi \le \frac{\pi}{2}.$$

   Prove that the intersection of the circular cylinder $x_1^2 + x_2^2 = 1$ with the plane $\mathbf{a} \cdot \mathbf{x} = 0$ is an ellipse. (Hint: consider the new basis $\mathbf{v}_1 = (-\sin\theta, \cos\theta, 0)^\top, \mathbf{v}_2 = (-\cos\phi\cos\theta, \sin\phi)^\top, \mathbf{v}_3 = \mathbf{a}$.)

4. Describe the projection of the cylindrical region $x_1^2 + x_2^2 = 1$, $-h \le x_3 \le h$ onto the general plane $\mathbf{a} \cdot \mathbf{x} = 0$.
5. Let $V = \mathrm{Span}\left((2,1,0,-2)^\top, (3,3,1,0)^\top\right) \subset \mathbb{R}^4$.

   (a) Find an orthogonal basis for V.

(b) Use your answer to part 1 to find the projection of $\mathbf{b} = (0, 4, -4, -7)^\top$ onto V.

(c) Use your answer to part 1 to find the projection matrix P_V.

6. (Direct sums) Let $W_1, ..., W_k$ be subspaces of a vector space V, such that

$$V = W_1 + ... + W_k.$$

Assume that

$$\begin{aligned} W_1 \cap W_2 &= \{\mathbf{0}\} \\ (W_1 + W_2) \cap W_3 &= \{\mathbf{0}\} \\ &\vdots \\ (W_1 + W_2 + ... + W_{k-1}) \cap W_k &= \{\mathbf{0}\}. \end{aligned}$$

Prove that V is the direct sum of the subspaces $W_1, ..., W_k$.