Lecture 4: Vector Spaces

Historic Notes

The ideas of linear combinations arose early in the study of differential equations. The history of the latter is itself a fascinating topic, with a great deal of activity beginning in the seventeenth century and continuing to the present day. The idea that a linear combination of solutions of a linear ordinary differential equation is itself a solution can be found in a 1739 letter from Leonhard Euler (1707-1783) to Johann Bernoulli (1667-1748). This means that the collection of all solutions forms a vector space. The notions of linear independence and basis also emerge in that letter, as Euler discusses writing the general solution of the differential equation as a linear combination of certain base solutions. These ideas continued to show up in works of other great mathematicians who studied differential equations, notably Jean le Rond d'Alembert (1717-1783) and Joseph-Louis Lagrange (1736-1813). The ideas of vector space and dimension has now been mostly credited to Hermann Gunther Grassman (1809-1877) for his study of geometry. In 1844 he published a seminal work describing his "calculus of extension," now called exterior algebra. His work inspired Giuseppe Peano (1858-1932) to make the modern definitions of basis and dimensions. The definition of an abstract vector space has originated in the 1888 publication *Geometrical Calculus*.

In 1877 Georg Cantor (1845-1918) made an amazing and troubling discovery: He proved that there is a one-to-one correspondence between points of $\mathbb{R}$ and points of $\mathbb{R}^2$. Although intuitively the plane is bigger than the line, Cantor's arguments says that they each have the same "number" of points. The actual definition of dimension that finally came about to resolve these issues leads us to a branch of mathematics known as topology. Cantor and Peano lay the groundwork for what is now the study of *fractals* or fractal geometry.

Four fundamental subspaces

Definition 1.0.1 (Nullspace). *Let A be an $m \times n$ matrix. The* nullspace *of A is the set of solutions of the homogeneous system $A\mathbf{x} = \mathbf{0}$:*

$$\mathbf{N}(A) = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}.$$

Definition 1.0.2 (Column space). *Let A be an $m \times n$ matrix with column vectors $\mathbf{a}_1, ..., \mathbf{a}_n \in \mathbb{R}^m$. We define the* column space *of A to be the subspace of $\mathbb{R}^m$ spanned by the column vectors:*

$$\mathbf{C}(A) = \mathrm{Span}(\mathbf{a}_1, ..., \mathbf{a}_n) \subset \mathbb{R}^m.$$

Proposition 1.0.3. *Let A be an $m \times n$ matrix. Let $\mathbf{b} \in \mathbb{R}^m$. Then $\mathbf{b} \in \mathbf{C}(A)$ if and only if $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. That is,*

$$\mathbf{C}(A) = \{\mathbf{b} \in \mathbb{R}^m : A\mathbf{x} = \mathbf{b} \textit{ is consistent}\}.$$

Proof. By definition, $\mathbf{C}(A) = \mathrm{Span}(\mathbf{a}_1, ..., \mathbf{a}_n)$, and so $\mathbf{b} \in \mathbf{C}(A)$ if and only if $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1, ..., \mathbf{a}_n$, i.e., $\mathbf{b} = x_1\mathbf{a}_1 + ... + x_n\mathbf{a}_n$ for some scalars $x_1, ..., x_n$. We conclude that $\mathbf{b} \in \mathbf{C}(A)$ if and only if $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x} \in \mathbb{R}^n$. Finally, the system $A\mathbf{x} = \mathbf{b}$ is consistent provided it has a solution. □

Definition 1.0.4 (Row space). *Let A be an $m \times n$ matrix with row vectors $\mathbf{A}_1, ..., \mathbf{A}_m \in \mathbb{R}^n$. We define the* row space *of A to be the subspace of $\mathbb{R}^n$ spanned by the row vectors $\mathbf{A}_1, ..., \mathbf{A}_m$:*

$$\mathbf{R}(A) = \mathrm{Span}(\mathbf{A}_1, ..., \mathbf{A}_m) \subset \mathbb{R}^n.$$

Noting that $\mathbf{R}(A) = \mathbf{C}(A^\top)$, it is natural to complete the rest as follows:

Definition 1.0.5 (Left nullspace). *We define the* left nullspace *of the $m \times n$ matrix A to be*

$$\mathbf{N}(A^\top) = \{\mathbf{x} \in \mathbb{R}^m : A^\top\mathbf{x} = \mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^m : \mathbf{x}^\top A = \mathbf{0}^\top\}.$$

Proposition 1.0.6. *Let A be an $m \times n$ matrix. Then $\mathbf{N}(A) = \mathbf{R}(A)^\perp$*

Proof. If $\mathbf{x} \in \mathbf{N}(A)$, then $\mathbf{x}$ is orthogonal to each row vector $A_1, ..., A_m$ of A. Then $\mathbf{x}$ is orthogonal to every vector in $\mathbf{R}(A)$ and is therefore an element of $\mathbf{R}(A)^\perp$. Thus $\mathbf{N}(A)$ is a subset of $\mathbf{R}(A)^\perp$, and so we need to show that $\mathbf{R}(A)^\perp$ is a subset of $\mathbf{N}(A)$. If $\mathbf{x} \in \mathbf{R}(A)^\perp$, this means that $\mathbf{x}$ is orthogonal to every vector in $\mathbf{R}(A)$, so $\mathbf{x}$ is orthogonal to each of the row vector $A_1, ..., A_m$. But this means that $A\mathbf{x} = \mathbf{0}$, so $\mathbf{x} \in \mathbf{N}(A)$. □

Since $\mathbf{C}(A) = \mathbf{R}(A^\top)$, when we substitute $A^\top$ for A, we have

Proposition 1.0.7. *Let A be an $m \times n$ matrix. Then $\mathbf{N}(A^\top) = \mathbf{C}(A)^\perp$.*

Proposition 1.0.8. *Let A be an $m \times n$ matrix. Then $\mathbf{C}(A) = \mathbf{N}(A^\top)^\perp$.*

Proof. Since $\mathbf{C}(A)$ and $\mathbf{N}(A^\top)$ are orthogonal subspaces, we have that $\mathbf{C}(A) \subset \mathbf{N}(A^\top)^\perp$. On the other hand, there is a system of constraint equations

$$\mathbf{c}_1 \cdot \mathbf{b} = ... = \mathbf{c}_k \cdot \mathbf{b} = 0$$

that give necessary and sufficient conditions for $\mathbf{b} \in \mathbb{R}^m$ to belong to $\mathbf{C}(A)$. Setting $V = \mathrm{Span}(\mathbf{c}_1, ..., \mathbf{c}_k) \subset \mathbb{R}^m$, this means that $\mathbf{C}(A) = V^\perp$. Since each such vector $\mathbf{c}_j$ is an element of $\mathbf{C}(A)^\perp = \mathbf{N}(A^\top)$, we conclude that $V \subset \mathbf{N}(A^\top)$. It follows that $\mathbf{N}(A^\top)^\perp \subset V^\perp = \mathbf{C}(A)$. Combining the two inclusions, we have $\mathbf{C}(A) = \mathbf{N}(A^\top)^\perp$. □

The following theorem summarizes geometric relations of the pairs of the four fundamental spaces.

Theorem 1.0.9. *Let A be an $m \times n$ matrix. Then*

1. $\mathbf{R}(A)^\perp = \mathbf{N}(A)$
2. $\mathbf{N}(A)^\perp = \mathbf{R}(A)$
3. $\mathbf{C}(A)^\perp = \mathbf{N}(A^\top)$
4. $\mathbf{N}(A^\top)^\perp = \mathbf{C}(A)$

Linear Independence and Basis

Given vectors $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathbb{R}^n$, it is natural to ask whether $\mathbf{v} \in \mathrm{Span}(\mathbf{v}_1, ..., \mathbf{v}_k)$. That is, do there exist scalars $c_1, .., c_k$ so that $\mathbf{v} = c_1\mathbf{v}_1 + ... + c_k\mathbf{v}_k$? This is in turn a question of whether a certain inhomogeneous system of linear equations has a solution. We are often interested in the question: is that solution unique?

Proposition 1.0.10. *Let $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathbb{R}^n$ and let $V = \mathrm{Span}(\mathbf{v}_1, ..., \mathbf{v}_k)$. An arbitrary vector $\mathbf{v} \in \mathrm{Span}(\mathbf{v}_1, ..., \mathbf{v}_k)$ has a unique expression as a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_k$ if and only if the zero vector has a unique expression as a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_k$; i.e.,*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k = \mathbf{0} \Rightarrow c_1 = ... = c_k = 0.$$

Proof. Suppose for some $\mathbf{v} \in V$ there are two different expressions

$$\mathbf{v} = c_1\mathbf{v}_1 + ... + c_k\mathbf{v}_k$$

and

$$\mathbf{v} = d_1\mathbf{v}_1 + ... + d_k\mathbf{v}_k.$$

Then subtracting, we obtain

$$\mathbf{0} = (c_1 - d_1)\mathbf{v}_1 + ... + (c_k - d_k)\mathbf{v}_k,$$

and so the zero vector has a nontrivial representation as a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_k$, since different expressions means that at least one difference $(c_i - d_i)$ is nonzero.

Conversely, suppose there is a nontrivial linear combination

$$\mathbf{0} = s_1\mathbf{v}_1 + ... + s_k\mathbf{v}_k.$$

Then, given any vector $\mathbf{v} \in V$, we can express $\mathbf{v}$ as a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_k$ in several ways: for instance, adding

$$\mathbf{v} = c_1\mathbf{v}_1 + ... + c_k\mathbf{v}_k$$

and

$$\mathbf{0} = s_1\mathbf{v}_1 + ... + s_k\mathbf{v}_k,$$

we obtain another formula for $\mathbf{v}$, namely,

$$\mathbf{v} = (c_1 + s_1)\mathbf{v}_1 + ... + (c_k + s_k)\mathbf{v}_k.$$

This completes the proof. □

This discussion leads us to make the following concept

Definition 1.0.11. *The set of vectors* $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ *is called* linear independent *if*

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k = \mathbf{0} \Rightarrow c_1 = ... = c_k = 0,$$

i.e., if the only way of expressing the zero vector as a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_k$ *is the trivial linear combination* $0\mathbf{v}_1 + ... + 0\mathbf{v}_k$.

Importantly, so if you are to prove a set of vectors $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is linear independent, you should write

Suppose $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k = \mathbf{0}$. I must show that $c_1 = ... = c_k = 0$.

Example 1.0.12. *We wish to decide whether the vectors*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \end{bmatrix} \in \mathbb{R}^4$$

for a linearly independent set.

Example 1.0.13. *Suppose* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. *Show that if* $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ *is linearly independent, the so is* $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{w}\}$.

Proposition 1.0.14. *Suppose* $\mathbf{v}_1, .., \mathbf{v}_k \in \mathbb{R}^n$ *form a linearly independent set, and suppose* $\mathbf{x} \in \mathbb{R}^n$. *Then* $\{\mathbf{v}_1, ..., \mathbf{v}_k, \mathbf{x}\}$ *is linearly independent if and only if* $\mathbf{x} \notin \mathrm{Span}(\mathbf{v}_1, ..., \mathbf{v}_k)$.

Proof. We can prove the contrapositive: Supposing that $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathbb{R}^n$ form a linearly independent set,

$$\{\mathbf{v}_1, ..., \mathbf{v}_k, \mathbf{v}\} \text{ is linearly dependent iff } \mathbf{v} \in \mathrm{Span}(\mathbf{v}_1, ..., \mathbf{v}_k).$$

Suppose that $\mathbf{v} \in \mathrm{Span}(\mathbf{v}_1, ..., \mathbf{v}_k)$. Then $\mathbf{v} = c_1\mathbf{v}_1 + ... + c_k\mathbf{v}_k$ for scalars $c_1, ..., c_k$, so

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k + (-1)\mathbf{v} = \mathbf{0},$$

which implies that $\{\mathbf{v}_1, ..., \mathbf{v}_k, \mathbf{v}\}$ is linearly dependent.
Now suppose $\{\mathbf{v}_1, ..., \mathbf{v}_k, \mathbf{v}\}$ is linearly dependent. This means that there are scalars $c_1, ..., c_k, c$, not all 0, such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c\mathbf{v} = \mathbf{0}.$$

Apparently, c cannot equal to 0, otherwise linearly independence of $\mathbf{v}_1, ..., \mathbf{v}_k$ would end up with an contraction. So dividing by c, we have

$$\mathbf{v} = -\frac{1}{c}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k$$

which tell us that $\mathbf{v} \in \mathrm{Span}(\mathbf{v}_1, ..., \mathbf{v}_k)$.

□

Definition 1.0.15. *Let* $V \subset \mathbb{R}^n$ *be a subspace. The set of vectors* $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ *is called a* basis *for* V *if*

i. $\mathbf{v}_1, ..., \mathbf{v}_k$ *span* V*; i.e.,* $V = \mathrm{Span}(\mathbf{v}_1, ..., \mathbf{v}_k)$, *and*
ii. $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ *is linearly independent.*

Example 1.0.16. *Let* $\mathbf{e}_1 = (1, ..., 0)^\top, \mathbf{e}_2 = (0, 1, 0, ..., 0)^\top, ..., \mathbf{e}_n = (0, ..., 0, 1)^\top \in \mathbb{R}^n$. *Then* $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ *is a basis for* $\mathbb{R}^n$, *called the* standard basis. *To check this, we must establish that properties (i) and (ii) hold for* $V = \mathbb{R}^n$.

Example 1.0.17. *Consider the plane given by* $V = \{\mathbf{x} \in \mathbb{R}^3 : x_1 - x_2 + 2x_3 = 0\} \subset \mathbb{R}^3$, *and two vectors*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Let c_1, c_2 be two real numbers,

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

which gives

$$\begin{bmatrix} c_1 - 2c_2 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So $\{\mathbf{v}_1, \mathbf{v}_2\}$ forms a basis of V.

Corollary 1.0.18. *Let $V \subset \mathbb{R}^n$ be a subspace, and let $\mathbf{v}_1, ...\mathbf{v}_k \in V$. Then $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is a basis for V if and only if every vector of V can be written uniquely as a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_k$.*

Definition 1.0.19. *When we write $\mathbf{v} = c_1\mathbf{v}_1 + ... + c_k\mathbf{v}_k$, we refer to $c_1, ..., c_k$ as the* coordinates *of $\mathbf{v}$ with respect to the ordered basis $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$.*

Example 1.0.20. *Consider the three vectors*

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Take a general vector $\mathbf{b} \in \mathbb{R}^3$, find a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Forming the augmented matrix and row reducing, we have

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 2 & 1 & 0 & b_2 \\ 1 & 2 & 2 & b_3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2b_1 - b_3 \\ 0 & 1 & 0 & -4b_1 + b_2 + 2b_3 \\ 0 & 0 & 1 & 3b_1 - b_2 - b_3 \end{array}\right]$$

Thus an arbitrary vector $\mathbf{b} \in \mathbb{R}^3$ can be written in the form

$$\mathbf{b} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3$$

where

$$\begin{aligned} c_1 &= 2b_1 - b_3 \\ c_2 &= -4b_1 + b_2 + 2b_3 \\ c_3 &= 3b_1 - b_2 - b_3. \end{aligned}$$

This process also gives a standard way of finding coordinates of $\mathbf{b}$ with respect to the basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

The following is an important fact.

Proposition 1.0.21. *Let A be an $n \times n$ matrix. Then A is nonsingular if and only if its column vectors form a basis for $\mathbb{R}^n$.*

Proof. Denote the column vectors of A by $\mathbf{a}_1, \mathbf{a}_2, ..., \mathbf{a}_n$. Using corollary above, we are to prove that A is nonsingular if and only if every vector in $\mathbb{R}^n$ can be written uniquely as a linear combination of $\mathbf{a}_1, ..., \mathbf{a}_n$. □

The following theorem tells us that every subspace has a basis.

Theorem 1.0.22. *Any subspace $V \subset \mathbb{R}^n$ other than the trivial subspace has a basis.*

Proof. Since $V \neq \{\mathbf{0}\}$, we can choose a nonzero vector $\mathbf{v}_1 \in V$. If $\mathbf{v}_1$ spans V, then we know $\{\mathbf{v}_1\}$ is a basis for V. If not, choose $\mathbf{v}_2 \notin \operatorname{Span}(\mathbf{v}_1)$. Previous proposition asserts that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. If $\mathbf{v}_1, \mathbf{v}_2$ span V, then $\{\mathbf{v}_1, \mathbf{v}_2\}$ will be a basis for V. If not, choose $\mathbf{v}_3 \notin \operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2)$. We know that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent and hence will form a basis for V if the three vectors span V. We continue in this fashion, and we are guaranteed that the process will terminate in at most n steps because once we have $n + 1$ vectors in $\mathbb{R}^n$, they must be linearly dependent. □

Dimensions

Proposition 1.0.23. *Let $V \subset \mathbb{R}^n$ be a subspace, let $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ be a basis for V, and let $\mathbf{w}_1, ..., \mathbf{w}_\ell \in V$. If $\ell > k$, then $\{\mathbf{w}_1, ..., \mathbf{w}_\ell\}$ must be linearly dependent.*

Proof. Each vector in V can be written uniquely as a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_k$. So let's write each vector $\mathbf{w}_1, ..., \mathbf{w}_\ell$ as such:

$$\begin{aligned}
\mathbf{w}_1 &= a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + ... + a_{k1}\mathbf{v}_k \\
\mathbf{w}_2 &= a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + ... + a_{k2}\mathbf{v}_k \\
&\vdots \\
\mathbf{w}_\ell &= a_{1\ell}\mathbf{v}_1 + a_{2\ell}\mathbf{v}_2 + ... + a_{k\ell}\mathbf{v}_k.
\end{aligned}$$

□

We can write

$$\begin{bmatrix} | & & | & & | \\ \mathbf{w}_1 & \dots & \mathbf{w}_j & \dots & \mathbf{w}_\ell \\ | & & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_k \\ | & | & & | \end{bmatrix} \begin{bmatrix} a_{11} & & a_{1j} & & a_{1\ell} \\ a_{21} & & a_{2j} & & a_{2\ell} \\ \vdots & \dots & \vdots & \dots & \vdots \\ a_{k\ell} & & a_{k\ell} & & a_{k\ell} \end{bmatrix}$$

where the jth column of the $k \times \ell$ matrix $A = [a_{ij}]$ consists of the coordinates of the vector $\mathbf{w}_j$ with respect to the basis $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$. We can write the above equation as

$$\begin{bmatrix} | & | & & | \\ \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_\ell \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ | & | & & | \end{bmatrix} A.$$

Since $\ell > k$, there cannot be a pivot in every column of A, and so there is a nonzero vector $\mathbf{c}$ satisfying

$$A \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_\ell \end{bmatrix} = \mathbf{0}.$$

Then we have

$$\begin{bmatrix} | & | & & | \\ \mathbf{w}_1 & \mathbf{w}_2 & \cdots & \mathbf{w}_\ell \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_\ell \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ | & | & & | \end{bmatrix} \left(A \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_\ell \end{bmatrix} \right) = \mathbf{0}.$$

That is, we have found a nontrivial linear combination of $\mathbf{w}_j$'s to be $\mathbf{0}$, which means $\{\mathbf{w}_j\}$ is linearly dependent.

Theorem 1.0.24. *Let $V \subset \mathbb{R}^n$ be a subspace, and let $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ and $\{\mathbf{w}_1, ..., \mathbf{w}_\ell\}$ be two basis for V. Then we have $k = \ell$.*

Proof. Since $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ forms a basis for V and $\{\mathbf{w}_1, ..., \mathbf{w}_\ell\}$ is known to be linearly independent, we use previous proposition to conclude that $\ell < k$. The same arguments applies to the other direction, so we have that $k \leq \ell$. Thus, $k = \ell$. □

Definition 1.0.25. *The* dimension *of a subspace $V \subset \mathbb{R}^n$ is the number of vectors in any basis for V. We denote the dimension of V by dim$V$. By convention,* $\dim \mathbf{0} = 0$.

Proposition 1.0.26. *Suppose V and W are subspaces of $\mathbb{R}^n$ with the property that $W \subset V$. If $\dim V = \dim W$, then $V = W$.*

Proof. Let $\dim W = k$ and let $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ be a basis for W. If $W \subsetneq V$, then there must be a vector $\mathbf{v} \in V$ with $\mathbf{v} \notin W$. Then $\{\mathbf{v}_1, ..., \mathbf{v}_k, \mathbf{v}\}$ is linearly independent, so $\dim V \geq k + 1$. This is a contradiction. Therefore, $V = W$. □

Dimensions of the Four Subspaces

In this section, we will specify a procedure for giving a basis for each of $\mathbf{R}(A), \mathbf{N}(A), \mathbf{C}(A), \mathbf{N}(A^\top)$. Their dimensions will follow immediately.

Theorem 1.0.27. *Let A be an $m \times n$ matrix. Let U and R, respectively, denote the echelon and reduced echelon form, respectively, of A, and write $EA = U$.*

1. *The (transposes of the) nonzero rows of U (or of R) give a basis for $\mathbf{R}(A)$.*

2. *The vectors obtained by setting each free variable equal to 1 and the remaining free variables equal to 0 in the general solution of $A\mathbf{x} = \mathbf{0}$ give a basis for $\mathbf{N}(A)$.*

3. *The pivot columns of A give a basis for $\mathbf{C}(A)$.*

4. *The rows of E that correspond to the zero rows of U give a basis for $\mathbf{N}(A^\top)$.*

The following results on dimension can be deduced.

Theorem 1.0.28. *Let A be an $m \times n$ matrix of rank r. Then*

1. $\dim \mathbf{R}(A) = \dim \mathbf{C}(A) = r$.

2. $\dim \mathbf{R}(A) = n - r$.

3. $\dim \mathbf{N}(A^\top) = m - r$.

Proof. There are r pivots and a pivot in each nonzero row of U, so $\dim \mathbf{R}(A) = r$. Similarly we have a basis vector for $\mathbf{C}(A)$ for each pivot, so $\dim \mathbf{C}(A) = r$. $\dim \mathbf{N}(A)$ is equal to the number of free variables, and this is the difference between the total number of variables, n, and the number of pivot variables, r. Lastly, the number of zero rows in U is the difference between the total number of rows, m, and the number of nonzero rows, r, so $\dim \mathbf{N}(A^\top) = m - r$. □

Corollary 1.0.29 (Nullity-Rank Theorem). *Let A be an $m \times n$ matrix. Then*

$$\textit{null}(A) + \textit{rank}(A) = n.$$

The following proposition is on the dimension of orthogonal complements.

Proposition 1.0.30. *Let $V \subset \mathbb{R}^n$ be a k-dimensional subspace. Then $\dim V^\perp = n - k$.*

Proof. Choose a basis $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ for V, and let these be the rows of a $k \times n$ matrix A. By construction, we have $\mathbf{R}(A) = V$. Notice that rank$(A) = \dim \mathbf{R}(A) = \dim V = k$. we have that $V^\perp = \mathbf{N}(A)$, so $\dim \mathbf{N}(A) = n - k$. □

As a consequence, we can prove the following

Theorem 1.0.31. *Let $V \subset \mathbb{R}^n$ be a subspace. Then every vector in $\mathbb{R}^n$ can be written uniquely as the sum of a vector in V and a vector in $V^\perp$. In particular, we have $\mathbb{R}^n = V + V^\perp$.*

Proof. Let $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ be a basis for $V^\perp$. Then we claim that the set $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is linearly independent. For suppose that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_n\mathbf{v}_n = \mathbf{0}.$$

Then we have

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k = -(c_{k+1}\mathbf{v}_{k+1} + ... + c_n\mathbf{v}_n)$$

Because only the zero vector can be in both V and $V^\perp$, we have

$$c_1\mathbf{v}_1 + ... + c_k\mathbf{v}_k = \mathbf{0} \text{ and } c_{k+1}\mathbf{v}_{k+1} + ... + c_n\mathbf{v}_n = \mathbf{0}.$$

Since each of the sets $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ and $\{\mathbf{v}_{k+1}, ..., \mathbf{v}_n\}$ is linearly independent, we conclude that $c_1 = ... = c_k = ...0$, as required. It follows that $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ gives a basis for an n-dimensional subspace of $\mathbb{R}^n$. Thus, every vector $\mathbf{x} \in \mathbb{R}^n$ can be written uniquely in the form

$$\mathbf{x} = (c_1\mathbf{v}_1 + ... + c_k\mathbf{v}_k) + (c_{k+1}\mathbf{v}_{k+1} + ... + c_n\mathbf{v}_n) = \text{element of } V + \text{element of } V^\perp$$

□

Homework

1. Suppose $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} \subset \mathbb{R}^3$ is linearly independent.

   (a) Prove that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) \neq 0$

   (b) Prove that $(\mathbf{u} \times \mathbf{v}, \mathbf{v} \times \mathbf{w}, \mathbf{w} \times \mathbf{u})$ is linearly independent.

2. Suppose $\mathbf{v}_1, ..., \mathbf{v}_k$ are nonzero vectors with the property that $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ whenever $i \neq j$. Prove that $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ is linear independent.

3. Let A be an $m \times n$ matrix and suppose $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathbb{R}^n$. Prove that if $\{A\mathbf{v}_1, ..., A\mathbf{v}_k\}$ is linearly indepdendent, then $\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ must be linearly independent.

4. Let A be an $n \times n$ matrix and suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^n$ are nonzero vectors that satisfy

$$\begin{aligned} A\mathbf{v}_1 &= \mathbf{v}_1 \\ A\mathbf{v}_2 &= 3\mathbf{v}_2 \\ A\mathbf{v}_3 &= 3\mathbf{v}_3 \end{aligned}$$

   Prove that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent. (Hint: start by showing that $\{\mathbf{v}_1, \mathbf{v}_2\}$ must be linearly independent.)

5. Find a basis for each of the given subspaces and determine its dimension.

   (a) $V = \mathrm{Span}\left((1,2,3)^\top, (3,4,7)^\top, (5,-2,3)^\top\right) \subset \mathbb{R}^3$

   (b) $V = \{\mathbf{x} \in \mathbb{R}^5 : x_1 = x_2, x_3 = x_4\} \subset \mathbb{R}^5$.

6. Give a basis for the orthogonal complement of each of the following subspaces of $\mathbb{R}^4$.

   (a) $V = \mathrm{Span}\left((1,0,3,4)^\top, (0,1,2,-5)^\top\right)$

   (b) $W = \{\mathbf{x} \in \mathbb{R}^4 : x_1 + 3x_3 + 4x_4 = 0, x_2 + 2x_3 - 5x_4 = 0\}$