Lecture 3: Theory of Linear Systems

We have seen in high school that we can view the unit circle $\{\mathbf{x} \in \mathbb{R}^2 : \|\mathbf{x}\| = 1\}$ either as the set of solutions of an equation or in terms of a parametric representation

$$\{\begin{bmatrix} \cos t \\ \sin t \end{bmatrix} : t \in [0, 2\pi)\}.$$

These are the implicit and explicit representations of this subset of $\mathbb{R}^2$. Similarly, any subspace $V \subset \mathbb{R}^n$ can be represented in two ways:

i. $V = \text{Span}\{\mathbf{v}_1, ..., \mathbf{v}_k\}$ for appropriate vectors $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathbb{R}^n$, this is the explicit or parametric representation;
ii. $V = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = 0\}$ for an appropriate $m \times n$ matrix A, this is the implicit representation, viewing V as the intersection of the hyperplanes defined by $A_i\mathbf{x} = 0$.

As we have seen the two expression of the unit circle, the implicit and explicit descriptions extends to even nonlinear settings.

Gaussian Elimination

In this section we give an explicit algorithm for solving a system of m linear equations in n vairables:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n &= b_m. \end{aligned}$$

With all the notations and concepts of matrix multiplication, we can write this in the form $A\mathbf{x} = \mathbf{b}$.

Some linear systems are easy to solve, but general complicated systems of equations require some algebraic manipulations before we can easily read off the general solution in parametric form. There are three basic operations we can perform on systems of equations that will not affect the solution set. They are called the *elementary operations*.

i. interchange any pair of equations;
ii. multiply any equation by a nonzero real number;
iii. replace any equation by its sum with a multiple of any other equation.

Example 1.0.1. *Consider the system of linear equations*

$$\begin{aligned} 3x_1 - 2x_2 + 2x_3 + 9x_4 &= 4 \\ 2x_1 + 2x_2 - 2x_3 - 4x_4 &= 6. \end{aligned}$$

After operation (i), we have

$$\begin{aligned} 2x_1 + 2x_3 - 2x_3 - 4x_4 &= 6 \\ 3x_1 - 2x_2 + 2x_3 + 9x_4 &= 4 \end{aligned}$$

Use operation (ii), multiplying the first equation by $\frac{1}{2}$, we have

$$\begin{aligned} x_1 + x_2 - x_3 - 2x_4 &= 3 \\ 3x_1 - 2x_2 + 2x_3 + 9x_4 &= 4 \end{aligned}$$

Use operation (iii), adding -3 times the first equation to the second, we have

$$\begin{aligned} x_1 + x_2 - x_3 - 2x_4 &= 3 \\ -5x_2 + 5x_3 + 15x_4 &= -5 \end{aligned}$$

Use operation (ii), multiplying the second equation by $-\frac{1}{5}$, we have

$$\begin{aligned} x_1 + x_2 - x_3 - 2x_4 &= 3 \\ x_2 - x_3 - 3x_4 &= 1 \end{aligned}$$

Finally, we use operation (iii), adding -1 times the second equaiton to the first, we have

$$x_1 + x_4 = 2$$
$$x_2 - x_3 - 3x_4 = 1.$$

It is easy to see that x_1 and x_2 are determined by x_3 and x_4. Thus the general solution of the system can be written as

$$\begin{aligned} x_1 &= 2 - x_4 \\ x_2 &= 1 + x_3 + 3x_4 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

which is equivalent to the vector form

$$\mathbf{x} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

which is a parametric representation of a plane in $\mathbb{R}^4$.

Before using elementary operations to solve a general system, we should make sure that performing elementary operations on a system of equations does not change its solutions.

Proposition 1.0.2. *If a system of equations $A\mathbf{x} = \mathbf{b}$ is changed into the new system $C\mathbf{x} = \mathbf{d}$ by elementary operations, then the systems have the same set of solutions.*

Proof. We need to show two results: 1. every solution of $A\mathbf{x} = \mathbf{b}$ is also a solution of $C\mathbf{x} = \mathbf{d}$, and 2. vice versa.

1. Start with a solution $\mathbf{u}$ of $A\mathbf{x} = \mathbf{b}$. Denoting the rows of A by $\mathbf{A}_1, ..., \mathbf{A}_m$, we have

$$\begin{aligned} \mathbf{A}_1 \cdot \mathbf{u} &= b_1 \\ &\vdots \\ \mathbf{A}_m \cdot \mathbf{u} &= b_m \end{aligned}$$

If we apply an elementary operation of type (i), $\mathbf{u}$ still satisfies precisely the same list of equations. If we apply an elementary operation of type (ii), i.e., multiplying the kth equation by $r \neq 0$. Note that if $\mathbf{u}$ satisfies $\mathbf{A}_k \cdot \mathbf{u} = b_k$, then it must satisfy

$$(r\mathbf{A}_k) \cdot \mathbf{u} = rb_k.$$

As for an elementary operation of type (iii), suppose we add r times the kth equation to the lth; since $\mathbf{A}_k \cdot \mathbf{u} = b_k$ and $\mathbf{A}_l \cdot \mathbf{u} = b_l$, it follows that

$$(r\mathbf{A}_k + \mathbf{A}_l) \cdot \mathbf{u} = (r\mathbf{A}_k \cdot \mathbf{u}) + (\mathbf{A}_l \cdot \mathbf{u}) = rb_k + b_l,$$

and so $\mathbf{u}$ satisfies the new lth equation.

2. To prove conversely that if $\mathbf{u}$ satisfies $C\mathbf{x} = \mathbf{d}$, the it satisfies $A\mathbf{x} = \mathbf{b}$, note that the inverse of elementary operation is an elementary operation.

□

We next introduce the *augmented matrix* for linear systems.

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

The elementary operations on a system become operations on the rows of the augmented matrix. Furthermore, we know that elementary operations do not change the solution set of a linear system, so we can freely perform elementary row operations on the augmented matrix of a system with the goal of finding an equivalent augmented matrix from which we can easily read off the general solution.

Definition 1.0.3. *We call the first* nonzero *entry of a row its* leading entry*. A matrix is in* echelon form *if*

1. *The leading entries move to the right in successive rows.*
2. *The entries of the column below each leading entry are all 0.*
3. *All rows of 0's are at the bottom of the matrix.*

A matrix is in reduced echelon form *if it is in echelon form and, in addition,*

4. *Every leading entry is 1.*
5. *All the entries of the column above each leading entry are 0 as well.*

If a matrix is in echelon form, we call the leading entry of any nonzero row a pivot*. We refer to the column in which a pivot appears as pivot columns and to the correpsonding variables as pivot variables. The remaining variables are called* free variables.

The word echelon origins from French, meaning "ladder". The standard steps of solving a linear system, or express its solution in terms of free variables consist of

- Reduce the augmented matrix of the linear system to echelon form
- Write the vector of unkowns $\mathbf{x} = (x_1, ..., x_n)$ in terms of free variables according to the reduced echelon form of augmented matrix.
- Express the vector of unknows or the general solution in *standard form*, i.e., the sum of a particular solution and a linear combination of vectors
- The particular solution is obtained by setting all the free variables equal to 0, and the vectors in the linear combination is obtained by letting the free variables 1 and 0 accordingly.

The above steps can be reviewed in the following example.

Example 1.0.4. *Suppose the augmented matrix of some linear system is*

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array}\right]$$

Note that this is already in reduced form. The general solution in terms of free variable is

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -4 \\ 0 \\ 2 \\ -1 \\ 1 \end{bmatrix}$$

Theory of Linear Systems: Existence, Constraint equations, and Rank

If $\mathbf{A}_1, ..., \mathbf{A}_m$ denote the row vectors of A, then the vectors $\mathbf{c}$ is a solution of $A\mathbf{x} = \mathbf{b}$ if and only if

$$\mathbf{A}_1 \cdot \mathbf{c} = b_1, ..., \mathbf{A}_m \cdot \mathbf{c} = b_m.$$

Geometrically, $\mathbf{c}$ is a solution exactly when it lies in the intersection of all the hyperplanes defined by the system of equations. On the other hand, we can define the column vectors as follows:

$$\mathbf{A}^j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

. Note that the matrix product $A\mathbf{x}$ can also be written as

$$A\mathbf{x} = x_1\mathbf{A}^1 + ... + x_n\mathbf{A}^n$$

Thus a solution $\mathbf{c}$ of the linear system $A\mathbf{x} = \mathbf{b}$ provides scalars $c_1, ..., c_n$ so that

$$\mathbf{b} = c_1\mathbf{A}^1 + ... + c_n\mathbf{A}^n$$

This gives us second geometric interpretation: A solution $\mathbf{c}$ gives a linear combination of $\mathbf{b}$ in terms of column vectors of A.

Definition 1.0.5. *If the system of equation $A\mathbf{x} = \mathbf{b}$ has no solutions, the system is said to be* inconsistent*; if it has at least one solution, then it is said to be* consistent*.*

Inconsistency from the echelon form of its augmented matrix: The system is inconsistent precisely when there is an equation of the form of

$$0x_1 + \ldots + 0x_n = c$$

for some nonzero scalar c. On the other hand, let $[U|\mathbf{c}]$ denote an echelon form of $[A|\mathbf{b}]$. Then the system is consistent iff any zero row in U corresponds to a zero entry in the vector $\mathbf{c}$.

There are two geometric way of consistency:

- $A\mathbf{x} = \mathbf{b}$ is consistent if and only if when the intersection of hyperplanes

$$\mathbf{A}_1\mathbf{x} = b_1, \ldots, \mathbf{A}_m\mathbf{x} = b_m$$

  is nonempty.
- or $\mathbf{b} \in \mathrm{Span}(\mathbf{A}^1, \ldots, \mathbf{A}^n)$.

So far we have studied linear combination of a set of vectors, existence of solution of linear systems, and intersection of hyperplanes. We will see that these three problems all lead to the concept of *rank* of a matrix.

Definition 1.0.6. *The* rank *of a matrix A is the number of nonzero rows (the number of pivots) in any echelon form of A. It is usually denoted by r.*

Then the number of rows of 0's in the echelon form is $m - r$, and $\mathbf{b}$ must satisfy $m - r$ constraint equations. Note that the rank of a matrix is well-defined, i.e., independent of the choice of echelon form. Given a system of m linear equations in n variables, let A denote its coefficient matrix and r the rank of A. The above property can be stated formally as follows.

Proposition 1.0.7. *The linear system $A\mathbf{x} = \mathbf{b}$ is consistent if and only if the rank of the augmented matrix $[A|\mathbf{b}]$ equals the rank of A. In particular, when the rank of A equals m, the system $A\mathbf{x} = \mathbf{b}$ will be consistent for all vectors $\mathbf{b} \in \mathbb{R}^m$.*

Proof. Let $[U|\mathbf{c}]$ denote the echelon form of the augmented matrix $[A|\mathbf{b}]$. We know that $A\mathbf{x} = \mathbf{b}$ is consistent if and only if any zero row in U corresponds to a zero entry in the vector $\mathbf{c}$, which occurs if and only if the number of nonzero rows in the augmented matrix $[U|\mathbf{c}]$ equals the number of nonzero rows in U, i.e., the rank of A. When $r = m$, there is no row of 0's in U and hence no possibility of inconsistency. □

Uniqueness and Nonuniqueness of Solutions

The theory of linear systems concerns how many solutions a given consistent system of equations has. Our previous study of solving linear systems suggests a connection between $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$.

Definition 1.0.8. *A system $A\mathbf{x} = \mathbf{b}$ is called* inhomogeneous *when $\mathbf{b} \neq \mathbf{0}$; the corresponding equation $A\mathbf{x} = \mathbf{0}$ is called the associated* homogeneous system*.*

Important! To related the solutions of the inhomogeneous system $A\mathbf{x} = \mathbf{b}$ and those of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$, we need the following fundamental algebraic observation.

Proposition 1.0.9. *Let A be an $m \times n$ matrix and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then*

$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}.$$

Proof. Ex. □

Theorem 1.0.10. *Assume the system $A\mathbf{x} = \mathbf{b}$ is consistent, and let $\mathbf{u}_1$ be a particular solution. Then all the solution are of the form*

$$\mathbf{u} = \mathbf{u}_1 + \mathbf{v}$$

for some solution $\mathbf{v}$ of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

Proof. Suppose $\mathbf{u}$ is an arbitrary solution of $A\mathbf{x} = \mathbf{b}$, then

$$A(\mathbf{u} - \mathbf{u}_1) = A\mathbf{u} - A\mathbf{u}_1 = \mathbf{b} - \mathbf{b} = \mathbf{0},$$

so $\mathbf{v} = \mathbf{u} - \mathbf{u}_1$ is a solution of the associated homogeneous system. The other direction is trivial. □

Proposition 1.0.11. *Suppose the system $A\mathbf{x} = \mathbf{b}$ is consistent. Then it has a unique solution if and only if the associated homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. This happens exactly when $r = n$.*

Definition 1.0.12. *An $n \times n$ matrix of rank $r = n$ is called* nonsingular. *An $n \times n$ matrix of rank $r < n$ is called* singular.

Let A be an $n \times n$ matrix. The following are equivalent:

1. A is nonsingular.
2. $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
3. For every $\mathbf{b} \in \mathbb{R}^n$, the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution.

Elementary Matrices and Inverses

So far we have focused on the interpretation of matrix multiplication in terms of columns, namely, the fact that the jth column of AB is the product of A with the jth column vector of B. Equally, the following is true

the jth row of AB is the product of the ith row vector of A with B.

Just as multiplying the matrix A by a column vector $\mathbf{x}$ on the right, which gives us the linear combination $x_1\mathbf{a}_1 + ... + x_n\mathbf{a}_n$ of the columns of A, we can easily check that multiplying A on the left by the row vector $[x_1 \dots x_m]$,

$$\begin{bmatrix} x_1 & x_2 & \dots & x_m \end{bmatrix} \begin{bmatrix} --- & \mathbf{A}_1 & --- \\ --- & \mathbf{A}_2 & --- \\ & \vdots & \\ --- & \mathbf{A}_m & --- \end{bmatrix}$$

yields the linear combination $x_1\mathbf{A}_1 + x_2\mathbf{A}_2 + ... + x_m\mathbf{A}_m$ of the rows of A.

We can perform row operations on a matrix A by multiplying on the left by appropriately chosen matrices. For example, if

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

then

$$E_1A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \\ 5 & 6 \end{bmatrix}, \quad E_2A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 20 & 24 \end{bmatrix} \quad E_3A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 5 & 6 \end{bmatrix}$$

Such matrices that give corresponding elementary row operations are called *elementary matrices*.

i. To interchange rows i and j, we should multiply by an elementary matrix of the form

$$\begin{matrix} \\ \\ i \;\rightarrow \\ \\ j \;\rightarrow \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & \dots & 0 & \dots & 1 & \dots & \\ & & & \ddots & & & \\ & \dots & 1 & \dots & 0 & \dots & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

- To multiply row i by a scalar c, we should multiply by an elementary matrix of the form

$$i \rightarrow \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & c & & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

iii. To add c times row i to row j, we should multiply by an elementary matrix of the form

$$\begin{matrix} \\ \\ i \rightarrow \\ \\ j \rightarrow \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & \ddots & & & \\ & \dots & c & \dots & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix}$$

Recall that if we want to find the constraint equations that a vector $\mathbf{b}$ must satisfy in order for $A\mathbf{x} = \mathbf{b}$ to be consistent, we reduce the augmented matrix $[A|\mathbf{b}]$ to echelon form $[U|\mathbf{c}]$ and set equal to 0 those entries of $\mathbf{c}$ corresponding to the rows of zeros in U. That is, when A is an $m \times n$ matrix of rank r, the constraint equations are merely the equations $c_{r+1} = ... = c_m = 0$. Letting E be the product of the elementary matrices corresponding to the elementary row operations required to put A in echelon form, we have $U = EA$ and so

$$[U|\mathbf{c}] = [EA|E\mathbf{b}].$$

That is, the constraint equation are the equations

$$\mathbf{E}_{r+1} \cdot \mathbf{b} = 0, \quad \dots, \quad \mathbf{E}_m \cdot \mathbf{b} = 0.$$

Here we can use equation $[U|\mathbf{c}] = [EA|E\mathbf{b}]$ to find a simple way to compute E: when we reduce the augmented matrix $[A|\mathbf{b}]$ to echelon form $[U|\mathbf{c}]$, E is the matrix so that $E\mathbf{b} = \mathbf{c}$.

Recall that the inverse of the $n \times n$ matrix A is the matrix A^{-1} satisfying $AA^{-1} = A^{-1}A = I_n$. It is convenient to have an inverse matrix if we wish to solve the system $A\mathbf{x} = \mathbf{b}$ for numerous vectors $\mathbf{b}$. If A is invertible, we can solve as follows:

$$A\mathbf{x} = \mathbf{b} \Rightarrow A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b} \Rightarrow (A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b} \Rightarrow \mathbf{x} = I_n\mathbf{x} = A^{-1}\mathbf{b}$$

In this computation, we see that if A is an invertible matrix, then $A\mathbf{x} = \mathbf{c}$ has unique solution for every $\mathbf{c} \in \mathbb{R}^n$, and so A must be nonsingular. We will see that the converse is also true. Suppose A is nonsingular, then every equation $A\mathbf{x} = \mathbf{c}$ has a unique solution. Consider equations

$$A\mathbf{x} = \mathbf{e}_j$$

for $j = 1, ..., n$, it follows from the nonsingularity of A that each equation has unique solution, say, $\mathbf{b}_j$, i.e.,

$$A\mathbf{b}_j = \mathbf{e}_j$$

and then we have

$$AB = A \begin{bmatrix} | & | & & | \\ \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ \mathbf{e}_1 & \mathbf{e}_2 & \dots & \mathbf{e}_n \\ | & | & & | \end{bmatrix} = I_n.$$

This means that the solution matrix B is the right inverse of A, but we still need $BA = I_n$ to assure that B is the left and right inverse of A. This can be understood by the process of Gaussian elimination in solving linear systems $A\mathbf{x} = \mathbf{e}_j$.

The Gaussian elimination is used to obtain the reduced echelon form $[I_n|B]$ from $[A|I_n]$, through a product E of elementary matrices, i.e.,

$$E[A|I] = [I|B],$$

so $EA = I$ and $B = E$, and then we have $BA = I$. Thus B is the inverse of A. This process has proved the following

Theorem 1.0.13. *An $n \times n$ matrix is nonsingular if and only if it is invertible.*

Example 1.0.14. *Determine the inverse of the matrix*

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & -2 & 2 \end{bmatrix}$$

HOMEWORK

1. Use elementary operation to find the general solution of the following system of equations.

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 + 2x_2 + x_3 &= 1 \\ x_2 + 2x_3 &= 1 \end{aligned}$$

2. For each of the following matrices A, determine its reduced echelon form and give the general solution of $A\mathbf{x} = \mathbf{0}$ in standard form.

(a)

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -2 & 3 & -1 \\ 3 & -3 & 0 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & -2 & 1 & 0 \\ 2 & -4 & 3 & -1 \end{bmatrix}$$

3. By solving a system of equations, find the linear combination of the vectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ that tives $\mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ -2 \end{bmatrix}$.

4. Decide whether $\mathbf{b}$ is a linear combination of

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

5. Find constraint equation that $\mathbf{b}$ must satisfy in order to be en element of

$$V = \text{Span}\left((-1, 2, 1), (2, -4, -2)\right)$$

6. Suppose A is an $m \times n$ matrix with rank m and $\mathbf{v}_1, ..., \mathbf{v}_k \in \mathbb{R}^n$ are vectors with $\text{Span}(\mathbf{v}_1, ..., \mathbf{v}_k) = \mathbb{R}^n$. Prove $\text{Span}(A\mathbf{v}_1, ..., A\mathbf{v}_k) = \mathbb{R}^m$.

7. Let A be an $m \times n$ matrix with column vectors $\mathbf{a}_1, ..., \mathbf{a}_n \in \mathbb{R}^m$.

   (a) Suppose $\mathbf{a}_1 + ... + \mathbf{a}_n = \mathbf{0}$. Prove rank$(A) < n$.

   (b) More generally, suppose that there is some linear combination $c_1\mathbf{a}_1 + ... + c_n\mathbf{a}_n = \mathbf{0}$, where some $c_i \neq 0$. Prove rank$(A) < n$.

8. Assume x_1, x_2, x_3 are distinct.

(a) Show that the matrix

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix}$$

is nonsingular.

(b) Show that the system of equations

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

always has a unique solution.

9. Let $A = \begin{bmatrix} 1 & \alpha \\ \alpha & 3\alpha \end{bmatrix}$.

(a) For which numbers α will A be singular?

(b) For all numbers α not on your list in part a, we can solve $A\mathbf{x} = \mathbf{b}$ for every vector $\mathbf{b} \in \mathbb{R}^2$. For each of the numbers α on your list, give the vectors $\mathbf{b}$ for which we can solve $A\mathbf{x} = \mathbf{b}$.