Lecture 2: Linear Transformations and Matrix Algebra

Historic Notes

Historically, the man who recognized the importance of the algebra of matrices and unified the various fragments of this theory into a subject worthy of standing by itself was Arthur Cayley (1821-1895).

Cayley was a British lawyer specializing in real estate law. He was successful but was known to say that the law was a way for him to make money so that he could pursue mathematics. He wrote about 300 mathematics papers during his fourteen years of practicing law. Finally, in 1863, he accepted a professorship at Cambridge University.

The term *matrix* was coined by Cayley's friend and colleague, James Joseph Sylvester (1814-1897). Many authors referred to what we now call a matrix as an "array" or "tableau."

Determinants first arose as an aid to solving equations. Although 2×2 determinants were implicit in the solution of a system of two linear equations in two unknowns given by Girolamo Cardano (1501-1576) in his work *Ars Magna* (1545), the Japanese mathematician Takakazu Seki (1642-1708) is usually credited with a more general study of determinants. In 1683 Seki published *Method of Solving Dissimulated Problems*, in which he studied determinants of matrices at least as large as 5×5. In the same year the German mathematician Gottfried Wilhelm von Leibniz (1646-1716) wrote a letter to Guillaume de l'Hôpital (1661-1704), in which he gave the vanishing of the determinant of a 3×3 system of linear equations as the condition for the homogeneous system to have a nontrivial solution.

After Seki and Leibniz, determinants were studied by many mathematicians including Gabriel Cramer (1704-1752), Étienne Bézout (1730-1783), Alexandre Vandermonde (1735-1796), Pierre-Simon Laplace (1749-1847).

The French mathematician Joseph-Louis Lagrange (1736-1813) seems to have been the first to notice the relationship between determinants and volume. Carl Friedrich Gauss (1777-1855) used matrices to study the properties of quadratic forms. Augustin Louis Cauchy also studied determinants in the context of quadratic forms. Finally, three papers that Carl Gustav Jacob Jacobi (1804-1851) wrote in 1841 brought general attention to the theory of determinants. He proved key results regarding the independence of sets of functions, inventing the Jacobian determinant which plays a major role in multivariable calculus.

Linear Transformations

We have learnt notations and concepts of functions and mappings, we are investigating linear mappings between vectors spaces.

Definition 1.0.1. *A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is called a* linear transformation *or* linear map *if it satisfies*

i. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ *for all* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$*;*

ii. $T(c\mathbf{v}) = cT(\mathbf{v})$ *for all* $\mathbf{v} \in \mathbb{R}^n$ *and scalars* c.

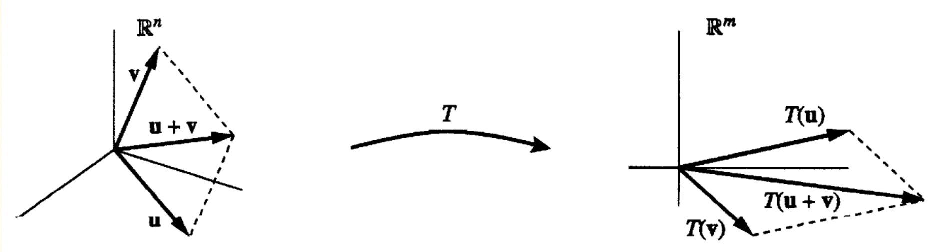
Consider linear combination of vectors, every vector in $\mathbb{R}^n$ can be written as a linear combination of the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The vectors $\mathbf{e}_1, ..., \mathbf{e}_n$ are often called the *standard basis* vectors for $\mathbb{R}^n$. Obviously, given the vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \text{we have } \mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + ... + x_n\mathbf{x}_n.$$

Back to linear transformation, if we think visually of T as mapping $\mathbb{R}^n$ to $\mathbb{R}^m$, then we have a diagram like following figure: The main point of the linearity properties is that the values of T on the standard



basis vectors $\mathbf{e}_1, ..., \mathbf{e}_n$ completely determine the function T: for suppose $\mathbf{x} = x_1\mathbf{e}_1 + ... + x_n\mathbf{e}_n \in \mathbb{R}^n$; then

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + ... + x_n\mathbf{e}_n) = x_1T(\mathbf{e}_1) + ... + x_nT(\mathbf{e}_n).$$

In particular, let

$$T(\mathbf{e}_j) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \in \mathbb{R}^m;$$

then to T we can naturally associate the $m \times n$ array

$$A = \begin{bmatrix} a_1 1 & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix},$$

which we call the *standard matrix* for T, and we will often denote this by $[T]$. To emphasize: the jth column of A is the vector in $\mathbb{R}^m$ obtained by applying T to the jth standard basis vector $\mathbf{e}_j$.

Example 1.0.2. *The most basic example of a linear map is the following. Fix* $\mathbf{a} \in \mathbb{R}^n$, *and define* $T : \mathbb{R}^n \to \mathbb{R}$ *by* $T(\mathbf{x}) = \mathbf{a} \cdot \mathbf{x}$. *By linearity of dot product, we have*

$$T(\mathbf{u} + \mathbf{v}) = a \cdot (\mathbf{u} + \mathbf{v}) = (\mathbf{a} \cdot \mathbf{u}) + (\mathbf{a} \cdot \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$$

and

$$T(c\mathbf{v}) = \mathbf{a} \cdot (c\mathbf{v}) = c(\mathbf{a} \cdot \mathbf{v}) = cT(\mathbf{v}),$$

moreover, it is easy to see that

$$\textit{if } \mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad \textit{then } [T] = [a_1 \ a_2 \ \dots \ a_n]$$

Example 1.0.3. *Consider the function* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *defined by rotating vectors in the plane counterclockwise by* $\pi/2$. *Then it is easy to see geometrically that*

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}.$$

Check the linearity of T.

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map, and let A be its standard matrix. We want to define the product of the $m \times n$ matrix A with the vector $\mathbf{x} \in \mathbb{R}^n$ is such a way that the vector $T(\mathbf{x}) \in \mathbb{R}^m$ is equal to $A\mathbf{x}$. In accordance with the formula of $T(\mathbf{x})$ in standard basis, we have

$$A\mathbf{x} = T(\mathbf{x}) = \sum_{i=1}^{n} x_iT(\mathbf{e}_i) = \sum_{i=1}^{n} x_i\mathbf{a}_i,$$

where

$$\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}, \ \mathbf{a}_2 = \begin{bmatrix} a_{12} \\ \vdots \\ a_{m2} \end{bmatrix}, \dots, \ \mathbf{a}_n = \begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix} \in \mathbb{R}^m$$

are the column vectors of the matrix A. That is, $A\mathbf{x}$ is the linear combination of the vectors $\mathbf{a}_1, ..., \mathbf{a}_n$, weighted according to the coordinates of the vector $\mathbf{x}$.

Algebra of Linear Mappings

Denote by $\mathcal{M}_{m\times n}$ the set of all $m \times n$ matrices. In an obvious way this set can be identified with $\mathbb{R}^{mn}$ (how?). We can add $m \times n$ matrices and multiply them by scalars, just as we did with vectors. For future references (no formal definition needed), we call a matrix *square* if $m = n$. We refer to the entries $a_{ii}, i = 1, ..., n$, as *diagonal* entries. We call the square matrix a *diagonal matrix* if $a_{ij} = 0$ whenever $i \neq j$.

A square matrix all of whose entries below the diagonal are 0 is called *upper triangular*; one all of whose entries above the diagonal are 0 is called *lower triangular*.

If $S, T : \mathbb{R}^n \to \mathbb{R}^m$ are linear maps and $c \in \mathbb{R}$, then we obviously form the linear maps $cT : \mathbb{R}^n \to \mathbb{R}^m$ and $S + T : \mathbb{R}^n \to \mathbb{R}^m$, defined, respectively, by

$$(cT)(\mathbf{x}) \overset{\text{def}}{=} c(T(\mathbf{x}))$$

$$(S+T)(\mathbf{x}) \overset{\text{def}}{=} S(\mathbf{x}) + T(\mathbf{x}).$$

The corresponding algebraic manipulations with matrices are clear: if $A = [a_{ij}]$, then cA is the matrix whose entries are ca_{ij}:

$$cA = c\begin{bmatrix} a_{11} & \dots & a_{1n} \\ a_{21} & \dots & a_{2n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{bmatrix} = \begin{bmatrix} ca_{11} & \dots & ca_{1n} \\ ca_{21} & \dots & ca_{2n} \\ \dots & \ddots & \dots \\ ca_{m1} & \dots & ca_{mn} \end{bmatrix}$$

Given two matrices A and $B \in \mathcal{M}_{m\times n}$, we define their *sum* entry by entry.

$$A + B = \begin{bmatrix} a_{11}+b_{11} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & \dots & a_{2n}+b_{2n} \\ \dots & \ddots & \dots \\ a_{m1}+b_{m1} & \dots & a_{mn}+b_{mn} \end{bmatrix}$$

Denote by O the zero matrix, any matrix added by O equals itself.

By the definition of matrix addition and scalar product (same for linear maps), we can check that scalar multiplication and addition satisfy the same properties as scalar multiplication and addition of vectors. We list them here for reference.

Proposition 1.0.4. *Let $A, B, C \in \mathcal{M}_{m\times n}$ and let $c, d \in \mathbb{R}$.*

1. $A + B = B + A$;
2. $(A + B) + C = A + (B + C)$;
3. $O + A = A$;
4. *There is a matrix $-A$ so that $A + (-A) = O$;*
5. $c(dA) = (cd)A$;
6. $c(A + B) = cA + cB$;
7. $(c + d)A = cA + dA$;
8. $1A = A$.

Recall that when $g(x)$ is the domain of f, we define $(f \circ g)(x) = f(g(x))$. So suppsoe we have linear maps $S : \mathbb{R}^{\prime} \to \mathbb{R}^n$ and $T : \mathbb{R}^n \to \mathbb{R}^m$. Then we can define $T \circ S : \mathbb{R}^p \to \mathbb{R}^m$ by $(T \circ S)(\mathbf{x}) = T(S(\mathbf{x}))$. Keep in mind that composition of functions is not commutative, so is linear maps. But composition is associative.

Matrix multiplication can be defined to correspond to the composition of linear maps. Let A be the $m \times n$ matrix representing T and let B be the $n \times p$ matrix representing S. We expect that the $m \times p$ matrix C representing $T \circ S$ can be expressed in terms of A and B. The jth column of C is the vector $(T \circ S)(\mathbf{e}_j) \in \mathbb{R}^m$. Now let

$$T(S(\mathbf{e}_j)) = T\begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{pmatrix} = b_{1j}\mathbf{a}_1 + b_{2j}\mathbf{a}_2 + ... + b_{nj}\mathbf{a}_n,$$

where $\mathbf{a}_1, ..., \mathbf{a}_n$ are the column vectors of A.

Above formulation can be made into the definition:

Definition 1.0.5. *Let A be an $m \times n$ matrix and B an $n \times p$ matrix. Their product AB is the $m \times p$ matrix whose jth column is the product of A with the jth column of B. That is, its ij-entry is*

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj},$$

i.e., the dot product of the ith row vector of A and the jth column vector of A, both of which are vectors in $\mathbb{R}^n$. Graphically, this calculation is illustrated as follows

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & & \vdots & \\ a_{i1} & a_{i1} & \cdots & a_{in} \\ & & \vdots & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & & b_{1j} & & b_{1p} \\ b_{21} & & b_{2j} & & b_{2p} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ b_{n1} & & b_{nj} & & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} \cdots & \cdots & \cdots \\ & \vdots & \\ \cdots & (AB)_{ij} & \cdots \\ & \vdots & \\ \cdots & \cdots & \cdots \end{bmatrix}$$

The jth column of AB is the product of A with the jth column vector of B.

Example 1.0.6. *Let*

$$A = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} \end{bmatrix}$$

Check that $A^2 = A$, so $A^n = A$ for all positive integer n. The geometric meaning of this operation is "projection onto a line". Note that

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} \\ \frac{2}{5} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

and

$$A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{2}{5} \\ \frac{4}{5} \end{bmatrix} = \frac{2}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

More generally,

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{x_1 + 2x_2}{5} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{\mathbf{x} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\|^2} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

which is exactly the projection of $\mathbf{x}$ onto the line spanned by $[1, 2]^\top$.

Proposition 1.0.7. *Let A and A' be $m \times n$ matrices; let B and B' be $n \times p$ matrices; let C be a $p \times q$ matrix, and let c be a scalar. Then*

1. *$AI_n = A = I_mA$. For this reason, I_n is called the $n \times n$ identity matrix.*
2. *$(A + A')B = AB + A'B$ and $A(B + B') = AB + AB'$. This is the distribution property of matrix multiplication over matrix addition.*
3. *$(cA)B = c(AB) = A(cB)$.*
4. *$(AB)C = A(BC)$. This is the associative property of matrix multiplication.*

Definition 1.0.8. *Let A be an $n \times n$ matrix. We say A is invertible if there is an $n \times n$ matrix B so that*

$$AB = BA = I_n.$$

We call B the inverse of the matrix A and denote this by $B = A^{-1}$.

Example 1.0.9. *Let*

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

Then $AB = I_n$ and $BA = I_n$, so B is the inverse of A.

Example 1.0.10. *For* 2×2 *matrix*

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

provided $ad - bc \neq 0$, *the inverse of* A *can be computed as*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Proposition 1.0.11. *Suppose* A *and* B *are invertible* $n \times n$ *matrices. Then their product* AB *is invertible, and*

$$(AB)^{-1} = B^{-1}A^{-1}.$$

The Transpose

The final matrix operation we will discuss is the *transpose*. When A is an $m \times n$ matrix with entries a_{ij}, the matrix $A^\top$ is the $n \times m$ matrix whose ij-entry is a_{ji}. We say a square matrix A is *symmetric* if $A^\top = A$ and *skew-symmetric* if $A^\top = -A$. These two important cases will be discussed in much detail in the future.

Proposition 1.0.12. *Let* A *and* A' *be* $m \times n$ *matrices, let* B *be an* $n \times p$ *matrix, and* c *be a scalar. Then*

1. $(A^\top)^\top = A$;
2. $(cA)^\top = cA^\top$;
3. $(A + A')^\top = A^\top + A'^\top$;
4. $(AB)^\top = B^\top A^\top$.

The transpose matrix will be important because of the interplay between dot product and transpose. If $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^n$, then

$$\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^\top \mathbf{y}.$$

Proposition 1.0.13. *Let* A *be an* $m \times n$ *matrix,* $\mathbf{x} \in \mathbb{R}^n$*, and* $\mathbf{y} \in \mathbb{R}^m$*. Then*

$$A\mathbf{x} \cdot \mathbf{y} = \mathbf{x} \cdot A^\top \mathbf{y}$$

Proof. Using the formula for the transpose of a product, we have

$$A\mathbf{x} \cdot \mathbf{y} = (A\mathbf{x})^\top \mathbf{y} = (\mathbf{x}^\top A^\top)\mathbf{y} = \mathbf{x}^\top (A^\top \mathbf{y}) = \mathbf{x} \cdot A^\top \mathbf{y}.$$

□

Determinants of 2×2 and 3×3 Matrices

Let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^2$ and consider the parallelogram P they span. The area of P is nonzero if and only if $\mathbf{x}$ and $\mathbf{y}$ are not collinear. The area of P is bh where $b = \|\mathbf{x}\|$ and $h = \|\mathbf{y}\| \sin\theta$, and we can calculate $\sin\theta$ from the formula

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \, \|\mathbf{y}\|}$$

Recall the ex on plane geometry:

Example 1.0.14. *If* $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, *set* $\rho(\mathbf{x}) = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$

(a) *Check that* $\rho(\mathbf{x})$ *is orthogonal to* $\mathbf{x}$. $\rho(\mathbf{x})$ *is obtained by rotating an angle* $\frac{\pi}{2}$ *counterclockwise.*

(b) *Give* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, *prove that* $\mathbf{x} \cdot \rho(\mathbf{y}) = -\rho(\mathbf{x}) \cdot \mathbf{y}$

If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1, y_2 \end{bmatrix}$, then we have from above exercise

$$\text{area}(P) = \rho(\mathbf{x}) \cdot \mathbf{y} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 y_2 - x_2 y_1.$$

The *signed area* of the parallelogram P to be the area of P when one turns counterclockwise from $\mathbf{x}$ to $\mathbf{y}$ and to be *negative* the area of P when one turns clockwise from $\mathbf{x}$ to $\mathbf{y}$. Then we have

$$\text{signed are}(P) = x_1y_2 - x_2y_1.$$

We consider the function

$$\mathcal{D}(\mathbf{x}, \mathbf{y}) = x_1y_2 - x_2y_1,$$

which is the function associates to each ordered pair of vectors $\mathbf{x}, \mathbf{y}$ the signed area of the parallelogram they span.

Some properties of $\mathcal{D}$.

Property 1. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, then $\mathcal{D}(\mathbf{y}, \mathbf{x}) = -\mathcal{D}(\mathbf{x}, \mathbf{y})$.

Algebraically, this is $y_1x_2 - y_2x_1 = -(x_1y_2 - x_2y_1)$.

Property 2. $\mathcal{D}(c\mathbf{x}, \mathbf{y}) = c\mathcal{D}(\mathbf{x}, \mathbf{y}) = \mathcal{D}(\mathbf{x}, c\mathbf{y})$.

Check $(cx_1)y_2 - (cx_2)y_1 = c(x_1y_2 - x_2y_1)$.

Property 3. $\mathcal{D}(\mathbf{x} + \mathbf{y}, \mathbf{z}) = \mathcal{D}(\mathbf{x}, \mathbf{z}) + \mathcal{D}(\mathbf{y}, \mathbf{z})$ and $\mathcal{D}(\mathbf{x}, \mathbf{y} + \mathbf{z}) = \mathcal{D}(\mathbf{x}, \mathbf{y}\mathcal{D}(\mathbf{x}, \mathbf{z})$.

Check that

$$(x_1 + y_1)z_2 - (x_2 + y_2)z_1 = (x_1z_2 - x_2z_1) + (y_1z_2 - y_2z_1)$$

Property 4. For the standard basis, $\mathcal{D}(\mathbf{e}_1, \mathbf{e}_2) = 1$.

The expression $\mathcal{D}$ is a 2×2 *determinant*, written $\begin{vmatrix} \mathbf{x} & \mathbf{y} \end{vmatrix}$. Indeed, given a 2×2 matrix A with column vectors $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^2$, we define

$$\det A = \mathcal{D}(\mathbf{a}_1, \mathbf{a}_2) = \begin{vmatrix} \mathbf{a}_1 & \mathbf{a}_2 \end{vmatrix}.$$

Given three vectors,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad \text{and } \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix},$$

we define

$$\mathcal{D}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \begin{vmatrix} | & | & | \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \\ | & | & | \end{vmatrix} = x_1 \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} - x_3 \begin{vmatrix} y_1 & z_1 \\ y_3 & z_3 \end{vmatrix} + x_3 \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}$$

Comparing to function $\mathcal{D}$ of two vectors, $\mathcal{D}$ of three vectors has similar properties. Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$ and c is a scalar, then

$$\mathcal{D}(\mathbf{x}, \mathbf{z}, \mathbf{y}) = -\mathcal{D}(\mathbf{x}, \mathbf{y}, \mathbf{z}),$$
$$\mathcal{D}(\mathbf{x}, c\mathbf{y}, \mathbf{z}) = c\mathcal{D}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathcal{D}(\mathbf{x}, \mathbf{y}, c\mathbf{z}),$$
$$\mathcal{D}(\mathbf{x}, \mathbf{y} + \mathbf{w}, \mathbf{z}) = \mathcal{D}(\mathbf{x}, \mathbf{y}, \mathbf{z}) + \mathcal{D}(\mathbf{x}, \mathbf{w}, \mathbf{z}).$$

To verify that $\mathcal{D}(\mathbf{z}, \mathbf{x}, \mathbf{z}) = -\mathcal{D}(\mathbf{x}, \mathbf{y}, \mathbf{z})$:

$$\begin{aligned}
\mathcal{D}(\mathbf{y}, \mathbf{x}, \mathbf{z}) &= y_1 \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} - y_2 \begin{vmatrix} x_1 & z_1 \\ x_3 & z_3 \end{vmatrix} + y_3 \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \\
&= y_1(x_2z_3 - x_3z_2) + y_2(x_3z_1 - x_1z_3) + y_3(x_1z_2 - x_2z_1) \\
&= -x_1(y_2z_3 - y_3z_2) + x_2(y_1z_3 - y_3z_1) - x_3(y_1z_2 - y_2z_1) \\
&= -x_1 \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} + x_2 \begin{vmatrix} y_1 & z_1 \\ y_3 & z_3 \end{vmatrix} - x_3 \begin{vmatrix} y_1 & z_2 \\ y_2 & z_2 \end{vmatrix} \\
&= -\mathcal{D}(\mathbf{x}, \mathbf{y}, \mathbf{z}).
\end{aligned}$$

Given two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, define a vector, called their *cross product*, by

$$\begin{aligned}
\mathbf{x} \times \mathbf{y} &= (x_2y_3 - x_3y_2)\mathbf{e}_1 + (x_3y_1 - x_1y_3)\mathbf{e}_2 + (x_1y_2 - x_2y_1)\mathbf{e}_3 \\
&= \begin{vmatrix} \mathbf{e}_1 & x_1 & y_1 \\ \mathbf{e}_2 & x_2 & y_2 \\ \mathbf{e}_3 & x_3 & y_3 \end{vmatrix},
\end{aligned}$$

where the latter is to be interpreted "formally". The geometric meaning of the cross product is the following proposition.

Proposition 1.0.15. *The cross product* $\mathbf{x}\times\mathbf{y}$ *of two vectors is orthogonal to both* $\mathbf{x}$ *and* $\mathbf{y}$ *and* $\|\mathbf{x}\times\mathbf{y}\|$ *is the area of the parallelogram* P *spanned by* $\mathbf{x}$ *and* $\mathbf{y}$*. Moreover, when* $\mathbf{x}$ *and* $\mathbf{y}$ *are nonparallel, the vectors* $\mathbf{x},\mathbf{y},\mathbf{x}\times\mathbf{y}$ *determine a parallelepiped of positive signed volume.*

Proof. Formula for the cross product gives

$$\mathbf{z}\cdot(\mathbf{x}\times\mathbf{y}) = \mathcal{D}(\mathbf{z},\mathbf{x},\mathbf{y}).$$

So $\mathbf{x}\cdot(\mathbf{x}\times\mathbf{y}) = 0$. Since $\mathcal{D}(\mathbf{x},\mathbf{y},\mathbf{x}\times\mathbf{y})$ is the signed volume of the parallelepiped spanned by $\mathbf{x},\mathbf{y}$, and $\mathbf{x}\times\mathbf{y}$, $\mathbf{x}\times\mathbf{y}$ is orthogonal to the plane spanned by $\mathbf{x}$ and $\mathbf{y}$, that volume is the product of the area of P and $\|\mathbf{x}\times\mathbf{y}\|$. On the other hand,

$$\mathcal{D}(\mathbf{x},\mathbf{y},\mathbf{x}\times\mathbf{y}) = \mathcal{D}(\mathbf{x}\times\mathbf{y},\mathbf{x},\mathbf{y}) = (\mathbf{x}\times\mathbf{y})\cdot(\mathbf{x}\times\mathbf{y}) = \|\mathbf{x}\times\mathbf{y}\|^2.$$

We can infer that

$$\|\mathbf{x}\times\mathbf{y}\| = \text{area}(P).$$

Homework

1. Let A be an $m\times n$ matrix. Show that $V = \{\mathbf{x}\in\mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$ is a subspace of $\mathbb{R}^n$.

2. Show that the matrix giving *reflection* across the line spanned by $\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$ is

$$R = \begin{bmatrix}\cos 2\theta & \sin 2\theta\\ \sin 2\theta & -\cos 2\theta\end{bmatrix}$$

3. Suppose A is a symmetric $n\times n$ matrix. Let $V\subset\mathbb{R}^n$ be a subspace with the property that $A\mathbf{x}\in V$ for every $\mathbf{x}\in V$. Prove that $A\mathbf{y}\in V^{\perp}$ for all $\mathbf{y}\in V^{\perp}$.

4. We say an $n\times n$ matrix A is *orthogonal* if $A^{\top}A = I_n$.

   a) Prove that the column vectors $\mathbf{a}_1,\ldots,\mathbf{a}_n$ of an orthogonal matrix A are unit vectors that are orthogonal to one another, i.e.,
   $$\mathbf{a}_i\cdot\mathbf{a}_j = 1 \text{ for } i=j,\quad 0 \text{ for } i\neq j.$$

   b) Prove that any 2×2 orthogonal matrix A must be of the form
   $$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \text{ or } \begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$$
   for some real number θ.

   c) Prove that if A is an orthogonal 2×2 matrix, then $\mu_A:\mathbb{R}^2\to\mathbb{R}^2$ is either a rotation or the composition of a rotation and a reflection.

5. Recall the definition of rotation matrix A_θ derived in class. Explain why $A_\theta^{-1} = A_\theta^{\top}$

6. Find the volume of the parallelepiped spanned by

$$\mathbf{x} = \begin{bmatrix}1\\2\\1\end{bmatrix},\quad \mathbf{y} = \begin{bmatrix}2\\3\\1\end{bmatrix},\quad \text{and } \mathbf{z} = \begin{bmatrix}-1\\0\\3\end{bmatrix}$$